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Characterization of derivations implemented by orthogonal projections in Hilbert Spaces

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Abstract

Let H^n be a finite dimensional Hilbert space and $\delta_{P,Q}$ be a generalized derivation induced by the orthogonal projections P and Q. In this study, we have approximated the norm of $\delta_{P,Q}$ by the formula $\| \delta_{P,Q} \| = \{\sum| \alpha |^2\}^{\frac{1}{2}} + \{\sum| \beta|^2\}^{\frac{1}{2}}$ and also showed that $\delta_{P,Q}$ is bounded and positive $\|\delta_{P,Q} \| \geq 0$ whenever P and Q are positive. Finally, we show compactness of $\delta_{P,Q}$ for compact operators P and Q.

Keywords: Orthogonal Projections, Hilbert Spaces, Matrix

1. Introduction

Studies have been done on generalized derivations, inner derivations, aspects of the underlying algebra $B(H)$ of these derivations and the structures of the operators inducing the derivations. An operator T is called D-symmetric, if the closure of the derivation δ_A is equal to the closure of the derivation δ_{T^*} in the norm topology. Anderson, Deddens and Williams [1] showed that for a trace class operator τ, $TP = PT$ implies that $T^*P = PT^*$ if T is D-symmetric operator. A generalization of this concept was used in [13] to define a class of pairs of operators $A, B \in B(H)$ say, such that $BT = TA$, implies that T^*p $= PT^*$, T^*, P^* being the adjoints of T and B respectively and T an element of trace class operators i.e. P-symmetric operators. Salah [14] constructed different C^*-algebras using the elements of P-symmetric operators i.e., $A, B \in B(H)$ such that $TA = AT$ implies that $A^*T = TB^*$. Indeed by [13], for $A, B \in B(H)$, if the pair (A, B) is generalized P-symmetric then: τ0(A, B), ι(A, B) and κ(A, B) are C^*-algebras w^*-closed in $B(H) \times B(H)$ and $\tau(A, B)$ is a bilateral ideal of $\iota(A, B)$. Continuity of derivations as mappings on different algebras is an important concept which has been fairly researched on. Kaplansky [8] and later Sakai [15], proved that a derivation δ of a C^*-algebra is automatically norm-continuous. This idea was later employed by Kadison [6] to show that such derivation is also continuous in the ultra-weak topology only if such a derivation is of an algebra of operators acting on a Hilbert space. Johnson [5] and later Sinclair [17] proved the automatic norm continuity of derivations of a semi-simple Banach algebra. Ringrose [12] used cohomological notation to prove that derivations from a C^*-algebra into a Banach-Module are automatically norm continuous, and that for appropriate class of dual algebra modules, they are continuous also relative to the ultraweak topology on the algebra and the weak $*$-topology on the module [12].

A linear mapping on an algebra X into an X-bimodal M is called a local derivation if for each $T \in A$, there is a derivation δ_T of X into M such that $\delta_T = \delta_T(T)$ [7]. Most of the studies on local derivations have been focused on finding the conditions which imply that a local derivation is a derivation. It is shown by Bresar [9] that in certain algebra, derivations can be characterized by some properties which local derivations trivially have, for example; Let X be a von Neumann algebra and let M be a normed X-bimodule. If a norm-continuous linear mapping δ of X into M is a local derivation, then δ is a derivation. A linear mapping T on a complex unital Banach algebra A is spectrally bounded if $r(Tx) \leq Mr(x)$ for all $x \in X$ and some $M \geq 0$ where $r(.)$ denotes the spectral radius [4]. Bresar [9] affirmed the fact that the image δ_X of an inner derivation δ of X is contained in the radical $radX$ of X if and only if δ is spectrally bounded, where $radX$ is the Jacobson radical. His argument was essentially based on the results due to Ptak [11], that a spectrally bounded inner derivation has the property that $\delta^2 X \subseteq \theta(X)$, the set of quasinilpotent elements of X. Curto [4] later on characterized the generalized inner derivations on a unital Banach algebra which are spectrally bounded. In particular, [4] simplified the argument due to [9], that every spectrally bounded inner derivation that maps into the radical is attainable [4]. Suppose $L(X,Y)$ is a space of all linear maps between Banach spaces X and Y, and S is a subset of $L(X,Y)$, a mapping $\Delta: X \mapsto T$ is said to be weak-2-local S map if for every $x, y \in X$ and $\phi \in Y^*$, there exists $T_{x,y,\phi} \in S$, depending on x, y and ϕ satisfying $\phi\Delta(x) = \phi T_{x,y,\phi}(x)$, and $\phi\Delta(y) = \phi T_{x,y,\phi}(y)$. The idea of weak-2-local derivations and automorphisms was introduced by Semrl [16] and explored extensively in [3] and [2]. In [10], Niazi and others proved that every weak-2-local derivation on a finite dimensional C^* −algebra is a linear derivation, and every weak2-local $*$-derivation on $B(H)$ is a linear $*$-derivation. It was then proved that every (weak)-2-local derivation on $C_0(L,A)$ is a linear derivation [3]. Consequently, [3] also showed that if B is an atomic von Neumann or a compact C^*-algebra, then every weak-2-local derivation on $C_0(L,B)$ is a linear derivation. Furthermore, for a general von Neumann algebra M, every 2-local derivation on $C_0(L,M)$ is a linear derivation.

We begin by applying the properties of orthogonal projections P and Q to construct a new orthogonal projection $P-Q$. We then proceed to apply these properties to give examples of the same on finite dimensional Hilbert space using matrices. We then construct a derivation $\delta_{P,Q}(X) = PX - XQ$ and show that $\delta_{P,Q}$ is a bounded linear operator which is continuous and positive. Finally, we calculate the norm $\| \delta_{P,Q} \|$ of the derivation $\delta_{P,Q}$and determine the norm and numerical radii inequalities for the same. In each of the properties of $\delta_{P,Q}$ discussed, we infer the results to the case when $P = Q$ to obtain the result for inner derivation δ_P. We shall denote set of all orthogonal projections acting on a Hilbert space H by $P_0(H)$.

Remark: The set of all derivations induced by orthogonal projections shall be denoted by $D_{op}[B(H)]$. Similarly, we shall denote by $D_{op}^I[B(H)]$ and D_{op}^G *[B(H)]* respectively the sets of all inner derivations and generalized derivations induced by orthogonal projections. It is noted that if $P = Q$ then $D_{op}^G[B(H)] \Leftrightarrow D_{op}^I[B(H)]$. Let H be a Hilbert space with a decomposition $H = V \oplus W^\perp$ where $W^\perp$ is the orthogonal compliment of W. Suppose that $P, Q \in P_0(H)$ are orthogonal projections on V and W respectively, then for any arbitrary linear operator *X*, there exists a new orthogonal projection $\delta_{PQ}(X) = (PX - XQ) \in D_{OP}[B(H)]$ which acts on the subspace $V \oplus W^\perp$.

2. Basic definitions

Definition 2.1: (52, Section 1). Let $B(H)$ be a C^*-algebra of all bounded linear operators on a Hilbert space H. An operator $T_{A,B} : B(H) \mapsto B(H)$ is called an elementary operator if it has the representation $T(X) = \sum_{i=1}^n A_i X B_i, \forall X \in B(H)$ where A_i, B_i are fixed in $B(H)$ or $M(H)$, the multiplier algebra of $B(H)$. For A and B fixed in $B(H)$, for all $X \in B(H)$ we define the particular elementary operators:

(i). the left multiplication operator (implemented by A) $L_A : B(H) \mapsto B(H)$ is defined by

$$L_A(X) = AX.$$

(ii). the right multiplication operator (implemented by B) $R_B : B(H) \mapsto B(H)$ is defined by

$$R_B(X) = XB.$$

(iii). the generalized derivation (implemented by A, B) $\delta_A : B(H) \mapsto B(H)$ is defined by

$$\delta_{A,B}(X) = AX - XB.$$

(iv). the inner derivation (implemented by A) $\delta_A : B(H) \mapsto B(H)$ is defined by

$$\delta_A(X) = AX - XA.$$

(v). the basic elementary operator (implemented by A, B)

$$M_{A,B}(X) = AXB$$

(vi). the Jordan elementary operators (implemented by A, B)

$$\mathcal{U}_{A,B}(X) = AXB + BXA, \forall X \in B(H).$$

Definition 2.2: Let $M_n(\mathbb{K})$ be a space of matrices over $\mathbb{K}$. The norm of $A \in M_n(\mathbb{K})$ is a function defined by $\| A \| = max\{\| A\bar{v} \|: \| \bar{v} \| = 1\}$ for a vector v which obeys all the norm properties and in addition, it is submultiplicative and subadditive i.e., $\| AB \| \leq \| A \|\| B \|$ and $\| A + B \| \leq \| A \| + \| B \|$ for $A, B \in M_n(\mathbb{K})$

Example 2.3: The following are some examples of the matrix (operator) norms:

(i). One-norm (the ℓ^1-norm) $\| T \|_1 = \sum_{j=1}^{n} | a_{ij} |$. Let $M_n(\mathbb{R}) : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be given by $T = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$thus for unit vectors $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $T(x_1) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $T(x_2) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ so $\| T(x_1) \| = \left\| \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\| = |1| + |3| = 4$ and $\| T(x_2) \| = \left\| \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\| = |2| + |1| = 3$ therefore $\| T \|_1 = 4$

(ii). Max-norm (the ℓ^∞-norm) $\| T \|^\infty = max | a_{ij} |$. Let T be as given in (i) above and vectors $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ then $T(x_1) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $T(x_2) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(iii). Two-norm (the ℓ^2-norm on T) $\| T \|_2 = (\sum_{j=1}^{n} |a_{ij}|)^{\frac{1}{2}}$.

Definition 2.4: (1, Definition 2.1). Let $T \in B(H)_1 \mapsto B(H)_2$ be a bounded linear operator and H_1,H_2 finite dimensional Hilbert spaces. The norm of the operator T is the smallest real number $\| T \|$ such that $\| Tx \| \leq \| T \|\| x \|$ where, $x \in H2$, i.e $\| T \| = sup\{\| Tx \|: \| x \| = 1\}$.

Remark 2.5: Given that $T \in B(H)$ is a compact operator, then we denote by $\{s_j(T)\}$, the singular values of T i.e the eigenvalues of $|T| = (TT^*)^{\frac{1}{2}}$. Schatten-p norm is an operator norm defined by $\| T \|_p = (\sum_{j=1}^{\infty} s_j^p(T))^{\frac{1}{p}}$ for 1 ≤ p ≤ ∞. For strictly positive p, the class of operators which admits the norms $\| T \|_p = (\sum_{j=1}^{\infty} s_j^p(T))^{\frac{1}{p}}$ are called Schatten-p operators and are denoted by C_p. C_p is an ideal in $B(H)$ of compact operators whose $\| T \|_p < \infty$, so that $\| |T|^2 \|_p = \| T \|_p^2$ for a finite p. The C_p class has two subclasses for $p = 1$ and $p = 2$ given by:

(i). Taxicab norm (C_1): For p = 1 then $\| T \|_1 = (\sum_{j=1}^{\infty} s_j(T))$ and $\| |T|^2 \|_{\frac{1}{2}} = \| T \|_1^2$. The class of all operators which admit the norm $\| T \|_1 = (\sum_{j=1}^{\infty} s_j(T))$ are called is called Trace class and is denoted by C_1

(ii). Hilbert-Schmidt norm (C_2): For $p = 2$ then $\| T \|_1 = (\sum_{j=1}^{\infty} s_j(T))^{\frac{1}{2}}$ and $\| |T|^2 \|^{\frac{1}{2}} = \| T \|_1^2$ The class of all operators which admit the norm $\| T \|_1 = (\sum_{j=1}^{\infty} s_j(T))^{\frac{1}{2}}$ are called is called Trace class and is denoted by C_2

Remark 2.6: The effect of an operator on a vector is a measure of how much an operator amplifies a norm of a unit vector. Operator norm $\| T \|$ is generally a vector norm on the range of the operator T such that $\| T^2 \| \leq \| T \|^2$. An operator acting on a finite dimensional Hilbert space can be represented by a matrix.

Definition 2.7: Suppose that $U, V, T \in B(H)$, where U and V are both unitary and A being compact, then a norm $|||\cdot|||$ defined by $|||UTV||| = |||T|||$ is called unitarily invariant norm.

Definition 2.8: Let H be a complex Hilbert space and T be a linear operator from H to itself. T is said to be positive if $\langle Tx, x \rangle \geq 0$, for all $x \in H$. This is denoted by $T \geq 0$ or $0 \leq T$. T is then said to be strictly positive or positive definite if $\langle Tx, x \rangle > 0$, for all $x \in H \backslash \{0\}$.

3. Results and discussions

We shall denote set of all orthogonal projections acting on a Hilbert space H by P0(H). In the sequel, we shall consider two decompositions of H^n thus; $H^n = H_1 \oplus H_2$ and $H^n = H_{11} \oplus H_{22}$ so that $H^n = H_1 \oplus H_2 = H_{11} \oplus H_{22}$

Lemma 3.0.1: Suppose there exist two distinct ways of decomposing H^n, $H^n = H_1 \oplus H_2$ and $H^n = H_{11} \oplus H_{22}$ and if $H_1 \subset H_{22}$ or $H_{11} \subset H_2$, then: $H^n = (H_1 \oplus H_{11}) \oplus (H_2 \cap H_{22})$:

Proof: Given that $H_1 \subset H_{22}$, then $H_1 + (H_2 \cap H_{22}) = (H_1 + H_2) \cap H_{22} = H^n \cap H_{22} = H_{22}$: and because $H_1 \cap (H_2 \cap H_{22}) = (H_1 \cap H_2) \cap H_{22} = \{0\}$, we have $H_{22} = H_1 \oplus (H_2 \cap H_{22})$.

Therefore: $Hn = H_{11} \oplus H_{22} = H_{11} \oplus H_1 \oplus (H_2 \cap H_{22}) = (H_1 \oplus H_{11}) \oplus (H_2 \cap H_{22})$: When H11 ⊂ H22, the same result follows by using $H_2 = H_{11} \oplus (H_2 \cap H_{22})$
We now give some examples to illustrate the construction of matrices of orthogonal projections.

Example 3.0.2: Find the matrix for the orthogonal projection $P : \mathbb{R}^3 \to W$ given that W is generated by the vectors $v_1 = (1,1,1)$ and $v_2 = (1,0,1)$.

To see this let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$,

$A^TA = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$ and $(A^TA)^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}$

Therefore, $Q = A(A^TA)^{-1}A^T$. $Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$ for any point $(x, y, z) \in \mathbb{R}^3$.

$$Q(x,y,z) = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left(\frac{x+z}{2}, y, \quad \frac{x+z}{2}\right).$$

Remark 3.0.3: Suppose that H^n is an n-dimensional Hilbert space, then $P(H^n)$ and $P_0(H^n)$ shall be used to denote the set of all projections acting on H^n and the set of all orthogonal projections acting on H^n respectively. Naturally, $P_0(H) \subset P(H) \subset B(H)$. The $B(H)$ used in this study is commutative. We begin by discussion of the properties of $P(H^n)$ and $P_0(H^n)$.

Theorem 3.0.4: Suppose that $P, Q \in P_0(H)$ onto H_1 and H_{11} respectively, then the following are equivalent:
(i). $P - Q$ is an orthogonal projection onto $H_{11} \cap H_1^{\perp}$.

(ii). $PQ = QP = P$.

(iii). $H_{22} \subset H_2$.

Proof:
(i). ⇒ (ii). Suppose that $P, Q \in P_0(H)$.
By the projection property, $(Q - P)^2 = Q - P \Rightarrow 2P = PQ + QP$.

Now, $Q(2P = PQ + QP) \Rightarrow 2QP = QPQ + Q^2P$
and $(2P = PQ + QP)Q \Rightarrow 2PQ = PQ2 + QPQ$
which means that $PQ = QP = P$.

(ii). ⇒ (iii).

For any $x \in H^n$, $Px \in H_1 \Rightarrow Px = QPx \in H_{11}$ which means that $H_1 \subset H_{11}$. Suppose $T_m = I_n - P_j\ (j = 1, 2, \ldots)$ then $PQ = P$ where $T_1 = I_1 - P,\ T_2 = I_2 - Q$ and $T_1T_2 = T_2$ therefore $T_2x \in H_{22} \Rightarrow T_2x = T_1T_2x \in H_2$ so that $H_{22} \subset H_2$. (iii). ⇒ (ii). Given that $H_{22} \subset H_2$, then for every $x \in H_n$, $Px \in H_1 \subset H_{11}$ which implies that $Q(Px) \Rightarrow QP = P$ and since $H_{22} \subset H_2$, then $T_2x \in H_{22} \subset H_2$ for $x \in H^n \Rightarrow T_1T_2x = T_2x = T_1T_2T_2 \Rightarrow (I_n - P)(I_n - Q) = (I_n - Q) \Rightarrow PQ = P$

(ii). ⇒ (i).
For $x \in (H_{11} \cap H_2)$, then $(Q - P)x = T_1Qx = T_1x = x$.
But suppose that $x = x_1 + x_2$
where $x_1 \in H_1$ and $x_2 \in H_{22}$
then $(Q-P)x = (Q-P)x_1 + (Q-P)x_2$
$= QT_1x_1 + T_1Qx_2 = 0.$
Therefore $(Q - P)$ is an orthogonal projection onto $H_{11} \cap H_2$ along $H_1 \oplus H_{22}$. $(H_2 = H_1^{\perp}$ and $H_{22} = H_{11}^{\perp})$. Now taking $X \in B(H)$, for a commutative $B(H)$, $X(P-Q) = XP - XQ = PX - XQ$ which is the desired derivation. Suppose that $p_n = \{f_i\}_{i=1}^k$ and $q_n = \{g_j\}_{j=1}^k$ are bases for H_1 and H_2 respectively with $H^n = H_1 \oplus H_2$ and $P: H^n \to H_1, Q: H^n \to H_2$ then, $\{f_i\}_{i=1}^k - \{g_j\}_{j=1}^k = \gamma$ is a basis for $H_{11} \cap H_1^{\perp}$ which is the range for P −Q.

Corollary 3.0.5: Let $P, Q \in P_0(H)$, thenhe operator $PX - XQ$ gives the shortest distance between $H_{11} \cap H_1^{\perp}$ and H^n

Proof: First we recall that $PX - XQ$ projects every point in H^n orthogonally to $H_{11} \cap H_1^{\perp}$. Let $x_1, x_2 \in (H_{11} \cap H_1^{\perp}) \subset H^n$, therefore for arbitrary $y \in H^n$, then $\| y - x_1 \|^2$, $\| y - x_2 \|^2 < dist(y, H_{11} \cap H_1^{\perp})^2 + \varepsilon$. Recall that $(H_{11} \cap H_1^{\perp})$ and $dist(H_{11} \cap H_1, H) = inf_{x' \in H_{11} \cap H_1^{\perp}} \| y - x' \|$.

So, by application of parallelogram law,
$\| x_1 - x_2 \|^2 = 2(\| y - x_1 \|^2 + \| y - x_2 \|^2) - 2 \| y - \frac{1}{2}(x_1 + x_2) \|^2$
$\leq 2\varepsilon.$

Therefore, there exists $t \in H_{11} \cap H_1^{\perp}$ $t = (P - Q)x$ for some $x \in H^n$ such that $\| y - t \| = dist(y, H_{11} \cap H_1^{\perp})$. So, the approximant of H^n to $H_{11} \cap H_1^{\perp}$ is the orthogonal projection $P - Q$. Therefore, every $y \in H$ can be uniquely written as $y = x + x'$ where $x' \in (H_{11} \cap H_1^{\perp})$ and $x \in (H_{11} \cap H_1^{\perp})$.

Lemma 3.0.6: Given a compact operator $X \in B(H)$, then PX and XQ are also compact for $P, Q \in P_0(H)$

Proof: Suppose that $X \in B(H)$ is compact and $Q \in P_0(H^n)$ then P is bounded. Let $x_n \in H^n$ be a bounded sequence. Then XQ is also bounded and contains a convergent subsequence. So XQ is compact. Now since X is compact, therefore Xx_n contains a convergent subsequence Xx_{n_k} which converges in the range of X. So PXx_{n_k} also converges.

Theorem 3.0.7: Suppose that $P, Q \in P_0(H^n)$ and a compact $X \in B(H)$, then $\delta_{P,Q}(X)$ is compact.

Proof: Let there exist bases p_n, q_n and b_n in H^n for P, Q and X respectively in H^n so that $(PX - XQ)$ takes the form, $\gamma = p_nb_n - b_nq_n$. Let γ be compact, U a closed unit ball of $(H_{11} \cap H_1^{\perp})$ and z_n a sequence of $\gamma(U)$. It suffices to show that there exists a subsequence of x_n that converges to U. By the supposition that γ is compact, for every $n \in \mathbb{N}$, $z_n = \gamma x_n$ and x_n belongs to the set U. So, there exists a subsequence x_{n_k} which converges weakly to $x \in U$. We show that γx_{n_k} converges to γx_n . Let γ_n be a sequence of finite rank operator that converges to γ. For any $m' \in \mathbb{N}$, $\gamma_{m'}$ is a closed set which is bounded in a finite dimensional subspace $(H_{11} \cap H_1^{\perp})$ of H^n, hence compact. So $\gamma_{m'}x_{n_k}$, $k \in \mathbb{N}$, converges to $\gamma_{m'}(x)$. Given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\| \gamma - \gamma_N \| < \frac{\epsilon}{3}$. Furthermore, given a fixed N, then $k' \in \mathbb{N}$, so that $\| \gamma_Nx_{n_k} - \gamma_Nx \| \leq \frac{\epsilon}{3}$ for $k \geq k'$.
So that:

$$\| \gamma_Nx_{n_k} - \gamma x \| \leq \| (\gamma - \gamma N)x_{n_k} \| + \| \gamma_N(x_{n_k} - x) \| + \| (\gamma_N - \gamma)x \|$$

$$\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$$

So, $z_{n_k} = \gamma x_{n_k}$ converges to $\gamma x \in \gamma(U)$ so that $\gamma(U)$ is compact. Suppose that $\gamma(U)$ is compact, then the union $\bigcup_{z_n \in \gamma(U)} B(z_n, \frac{1}{n})$ is an open covering of the compact set $\gamma(U)$, and therefore we can obtain vectors $x_1^{(n)}, x_2^{(n)}, \ldots, x_k^{(n)} \in H^n$ such that $\bigcup_{i=1}^{k} U(\gamma x_i^{(n)}, \frac{1}{n})$ is a covering of $\gamma(U)$. Suppose that H^n span $\gamma x_i^{(n)}, i \in \mathbb{N}, T$ an arbitrary orthogonal projection on H^n.

Let also $\gamma_n = T_n \gamma$.

For $\epsilon > 0$, and $N > \frac{1}{\epsilon}$ if $n \geq N$ with $\| x \| \leq 1$, then $\| \gamma x - \gamma_n x \| = \| \gamma x - T\gamma x \|$.

Now because $T\gamma \in \gamma_n$ is the point in H^n closest to γx, therefore

$$\| \gamma x - \gamma_n x \| \leq inf_{1 \leq i \leq n} \| \gamma x - \gamma x_i^{(n)} \| < \frac{1}{n} < \epsilon.$$

So, $\gamma_n \to \gamma$ as $n \mapsto \infty$. Which implies that $D_{op}^{G} [B(H)]$ is a compact ideal of $B(H)^n$. Now since $\gamma_n \to \gamma \in D_{op}^{G} [B(H)]$ the assertion is proved.

Example 3.0.8: Let $H^n = \ell^2$ and $m \times m$ operators $P = [a_{ij}]$ and $Q = [b_{ij}]$

such that

$$a_{ij} = \begin{cases} p_n, & i = j \\ 0, & i \neq j \end{cases} \text{ and } b_{ij} = \begin{cases} q_n, & i = j \\ 0, & i \neq j \end{cases}$$

for $n = m - 1, \; q_n^* (q_n - 1) = 0$ and $m \geq 2$.

Let the operators P and Q be bounded i.e. for $n \geq 1$, $p_n = (p_1, p_2, \ldots) \in \ell^\infty$ and $q_n = (q_1, q_2, \ldots) \in \ell^\infty$. Let P be majorized by Q or Q majorized by P so that $(p_n - q_n)$ is also diagonal and $(p_n - q_n) = ((p_1 - q_1), (p_2 - q_2), \ldots) \in \ell^\infty$ Suppose that $lim_{n\to\infty}(p_n - q_n) = 0$ and $(P - Q)_n = diag((p_1 - q_1), (p_2 - q_2), 0, 0, \ldots)$, then $(P - Q)_n$ is compact and $\| (P - Q) - (P - Q)_n \| = sup\{|p_n - q_n| \geq n + 1\} \to 0$. For an arbitrary $X \in B(\ell)^2$, then $PX - XQ$ is also compact.

Suppose that $x_n \in \ell^2$, with the following conditions, $\| x_n \| \leq 1, \; \| (PX - XQ)x \| = \| PX - XQ \|$ and some $\alpha, \beta \in \mathbb{F}$, such that $lim_{n\to\infty}\langle (p_n - q_n)x_n, x_n \rangle = lim_{n\to\infty}\langle (p_n - q_n)^2 x_n, x_n \rangle$

$$= lim_{n\to\infty}\langle (p_n - q_n)^* x_n, x_n \rangle$$

$$= |\alpha - \beta|.$$

Thus $|\alpha - \beta| \in \mathbb{R}^+$.

Example 3.0.9: Let $H^n = L^2(\mathbb{T})$ be the space of 2π–periodic functions and a constant function $u = \frac{1}{\sqrt{2\pi}}$, with $\| u \| = 1$, then the orthogonal projections Pu and Qu are defined by $Puf = \frac{1}{2\pi} \int_0^{2\pi} f(x)dx$ and $Quf = \frac{1}{2\pi} \int_0^{2\pi} g(x)dx$. So $P - Q = \frac{1}{2\pi} \int (f(x) - g(x))dx$ and so for $X \in B(L^2(\mathbb{T}))$, then $PX - XQ = \frac{1}{2\pi} \int_0^{2\pi} b(x)(f(x) - g(x))dx$ is compact.

We apply the following example in showing how a matrix of $\delta_{P,Q}$ can be constructed.

Example 3.1.0: Consider two sets of vector $v_1 = (0, 1, 0), \; v_2 = (0, 1, 1)$ and $u_1 = (1, 1, 1), \; u_2 = (1, 0, 1)$.
By simple calculation, we get that

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \; A^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \; A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \; (AA^T)^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

So $A(A^T A)^{-1} A^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ thus we get an orthogonal projection

$$P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Similarly for the second set of vectors, we get another orthogonal projection

$$XQ = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

Now for an arbitrary operator with a matrix representation

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ then } PX = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{and } XQ = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \text{so that } PX - XQ = \begin{bmatrix} \frac{-1}{2} & 0 & \frac{-1}{2} \\ \frac{-1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}.$$

Example 3.1.1: Let H^n be a complex four-dimensional Hilbert space and $B(H)$ algebra of 4×4 matrices. We take $P_0(H)$ to be the subalgebra of diagonal matrices, so $\delta_{P,Q} : P_0(H) \to B(H)$. Suppose that $P, Q \in P_0(H)$are selfadjoint orthogonal projections onto H_1 and H_{11} spanned by the orthogonal unit vectors

$$x_1 = \frac{1}{2\sqrt{3}} \left(1,\ -1 + i\sqrt{3},\ \frac{1}{\sqrt{14}}(4 - 5i\sqrt{3}),\ \frac{1}{\sqrt{14}}(-2 + i\sqrt{3})\right)$$

$$x_2 = \frac{1}{2\sqrt{3}} \left(1,\ -1 - i\sqrt{3},\ \frac{1}{\sqrt{14}}(-2 - i\sqrt{3}),\ \frac{1}{\sqrt{14}}(4 + 5i\sqrt{3})\right)$$

and the unit vector

$$x_3 = \frac{1}{2\sqrt{3}} \left(1,\ 2,\ \sqrt{\tfrac{7}{2}},\ \sqrt{\tfrac{7}{2}}\right)$$ respectively.

Then for an arbitrary operator $X \in B(H)$ with $\| X \| = 1$, operator $PX - XQ$ has a Hermitian matrix
Given by

$$\frac{1}{12}\begin{bmatrix} \begin{matrix} 1 & -4 \\ -4 & 4 \end{matrix} & \begin{matrix} \frac{1}{\sqrt{14}}(-5 + 6i\sqrt{3}) \\ -2\sqrt{14} \end{matrix} & \begin{matrix} \frac{1}{\sqrt{14}}(-5 - 6i\sqrt{3}) \\ -2\sqrt{14} \end{matrix} \\ \frac{1}{\sqrt{14}}(-5 - 6i\sqrt{3}) \quad -2\sqrt{14} & \frac{7}{2} & \frac{1}{14}(-95 + 12i\sqrt{3}) \\ \frac{1}{\sqrt{14}}(-5 + 6i\sqrt{3}) \quad -2\sqrt{14} & \frac{1}{14}(-95 - 12i\sqrt{3}) & \frac{7}{2} \end{bmatrix}$$

which is also idempotent. We now consider the linearity of $\delta_{P,Q}$ in the following proposition.

Proposition 3.1.2: A derivation $PX - XQ$ is linear for an arbitrary $X \in B(H)$.

Proof: Let $\{f_i\}_{i\in I}$ and $\{q_i\}_{i\in J}$ be two orthonormal bases for $H_{11} \cap H_1^{\perp}$ and $(H_{11} \cap H_1^{\perp})^{\perp}$, respectively, and $T = I - (P - Q)$ be the orthogonal projection on $(H_{11} \cap H_1^{\perp})^{\perp}$. Suppose $X \in B(H)$ then for $x_1, x_2 \in H$ and $\alpha, \beta \in \mathbb{K}$, then by theorem 4.12, there exist $\gamma = (p_n b_n - b_n q_n)$ such that

$$\begin{aligned} \gamma(\alpha x_1 + \beta x_2) &= \gamma \textstyle\sum_{i\in I} \sum_n \langle \alpha x_1 + \beta x_2, f_i \rangle \\ &= \textstyle\sum_{i\in I} \sum_n \langle (p_n b_n - b_n q_n)(\alpha x_1 + \beta x_2), f_i \rangle \\ &= \textstyle\sum_{i\in I} \sum_n \langle \alpha p_n b_n x_1 + \beta p_n b_n x_2 - \alpha b_n q_n x_1 - \beta b_n q_n x_2, f_i \rangle \\ &= \alpha \textstyle\sum_{i\in I} \sum_n \langle p_n b_n x_1, f_i \rangle + \beta \sum_{i\in I} \sum_n \langle p_n b_n x_2, f_i \rangle - \alpha \sum_{i\in I} \sum_n \langle b_n q_n x_1, f_i \rangle - \beta \sum_{i\in I} \sum_n \langle b_n q_n x_2, f_i \rangle \\ &= \alpha \textstyle\sum_{i\in I} \sum_n \langle (p_n b_n - b_n q_n) x_1, f_i \rangle + \beta \sum_{i\in I} \sum_n \langle (p_n b_n - b_n q_n) x_2, f_i \rangle \end{aligned}$$

$= \alpha\gamma x_1 + \beta\gamma x_2$

Corollary 3.1.3: A derivation $PX - XQ$ is linear on (i) H^n and (ii) $B(H^n)$.

Proof: (i). Linearity in H^n: Given that $P, Q \in P_0(H)$, by [76, lemma 2] we can obtain a pair $x, y \in H$ and $\alpha, \beta \in \mathbb{K}$ and on setting $\lim_n \| Px_n \| \Rightarrow \| P \|$, $\lim_n \| Qx_n \| \Rightarrow \| Q \|$ and $\lim_n \langle Qx_n, x_n \rangle \to |\mu|$, $\lim_n \langle Px_n, x_n \rangle \to |\lambda|$ and also setting $Qx = \alpha x + \beta y$, $Px = \alpha^* x + \beta^* y$ with $\langle x, y \rangle = 0$ and $\| x \| = \| y \| = 1$. Set also that $Xx = x$, $Xy = -y$ and also that X acts on $\{x, y\}$ then, $P - Q$ is an orthogonal projection onto $H_{11} \cap H_1^{\perp}$. On respective post and premultiplication of P and Q of $P - Q$ by X gives a new operator of the form γ such that for $x \in H$,

$$\gamma x = Px - \alpha^* Xx + \beta^* Xy$$

$$= \alpha x + \beta y - \alpha^* x + \beta^* y$$

$$= (\alpha - \alpha^*)x + (\beta + \beta^* y, \text{ for } \alpha x, \beta y \in H^n$$

and on the other hand

$$\delta_{P,Q}(X)(\alpha x + \beta y) = \gamma(\alpha x + \beta y)$$

$$= \gamma\alpha x + \gamma\beta y$$

$$= \alpha\delta_{P,Q}(X)x + \beta\delta_{P,Q}(X)y$$

(ii). Linearity in $B(H^n)$: By similar calculation, we can find a pair of operators $X, Y \in B(H)$ and $\alpha, \beta \in \mathbb{K}$

$$\delta_{P,Q}(\alpha X + \beta Y)x = \{P(\alpha X + \beta Y) - (\alpha X + \beta Y)Q\}x$$

$$= \{\alpha PX + \beta PY - \alpha XQ - \beta YQ\}x$$

$$= \alpha PX - \alpha XQ + \beta PY - \beta YQx$$

$$= \{\alpha\delta_{P,Q}(X)x + \beta\delta_{P,Q}(Y)\}x.$$

So, $\delta_{P,Q}X$ is linear on both H^n and $B(H)$.

Proposition 3.1.4: Suppose that $P, Q \in P0(H)$ and an arbitrary $X \in B(H)$, then $\delta^2{}_{P,Q}(X) = PX + XQ - 2PXQ$.

Proof:

$$\delta 2P, Q(X) = \delta P, Q(PX - XQ)$$

$$= P(PX - XQ) - (PX - XQ)Q$$

$$= PPX - PXQ - PXQ + XQQ$$

$$= PPX + XQQ - 2PXQ$$

$$= PX + XQ - 2PXQ$$

Theorem 3.1.5: Suppose that $P, Q \in P_0(H)$, then the derivation $\delta_{P,Q}(X) = PX - XQ$ is bounded from below.

Proof: By the definition of $\delta_{P,Q}$ we observe that $\delta_{P,Q}(X) = p_n - q_n$ is meaningful for the bases p_n and q_n of P and Q respectively with $\sum_n \| f_n \|^2 = 1$. Since p_n and q_n are bounded, from the definition of $\delta_{P,Q}(X)$, we have,

$$\| \delta_{P,Q}(f_n) \|^2 = \| \textstyle\sum_n (p_n f_n - f_n q_n) \|^2$$

$$\geq \textstyle\sum_n \| p_n f_n \|^2 - \sum_n \| f_n p_n \|^2$$

$= \{\sum_n |p_n|^2 - \sum_n |q_n|^2\} \| f_n \|^2$.

Since the difference of finite summation of p_n and q_n is also bounded, by taking supremum of both sides of the above inequality gives $\| \delta \| \geq \{\sum_n |p_n|^2\}^{1/2} - \{\sum_n |q_n|^2\}^{\frac{1}{2}}$.

Proposition 3.1.6: Suppose that $P, Q \in P_0(H)$, then the derivation $\delta_{P,Q}$ is bounded from above.

Proof: Let P, Q and X be induced by p_n, q_n respectively and f_n as arbitrary elements of $B(H)$. By the definition of δ, we have for $\{\| \sum_n f_n \|^2\} = 1$ and $\| \sum_n f_n \| \leq 1$ that

$$\| \delta_{P,Q} \|^2 = \| \sum_n (p_n f_n - f_n q_n) \|^2$$

$$\leq \| \sum_n p_n f_n \|^2 + \| \sum_n f_n q_n \|^2$$

$$\leq \{\sum_n |p_n|^2 + \sum_n |q_n|^2\}\{\| \sum_n f_n \|^2\}$$

$$= \{\sum_n |p_n|^2 + \sum_n |q_n|^2\}$$

$$= \{\sum_n |p_n| + \sum_n |q_n|\}.$$

Taking the supremum of both sides of the inequality gives us $\| \delta_{P,Q}(X) \| \leq \{\sum_n |p_n|\}^{\frac{1}{2}} + \{\sum_n |q_n|\}^{\frac{1}{2}}$.

Proposition 3.1.7: Suppose that $P, Q \in P_0(H)$, then $\delta_{P,Q}(X)$ has a bounded inverse on $H_{11} \cap H_1^{\perp}$ if and only if $\delta_{P,Q}(X)$ is bounded from below.

Proof: From the definition of $\delta_{P,Q}$, we have that $\delta_{P,Q}(X)$ is a transformation $\delta_{P,Q}(X)$: $H^n \to H_{11} \cap H_1^{\perp}$. Now suppose that p_n, q_n and x_n as described in theorem 4.12 are all bounded from, then so is γ, and therefore there exists a real number $m > 0$ such that $\| \gamma(x) \| \geq m \| x \| \;\; \forall x \in H^n$. This means that γ is a one-to-one map. Thus γ is a bijection and hence has an inverse, $\gamma^{-1} : B(H) \to P_0(H)^n$ which is linear and onto. We then show that γ^{-1} is bounded and $\| \gamma^{-1} \| \leq \frac{1}{m}$.
Let $y \in H_{11} \cap H_1^{\perp}$ and $P', Q' \in P_0(H)^n$, then $y \in \gamma(x)$, for unique elements $x \in H^n$ and $P, Q \in P_0(H)^n$. Now, since γ is bounded from below, we get $m^{-1} \| y \| \geq \| \gamma^{-1} y \|$ i.e., $\| \gamma^{-1} y \| \leq m^{-1} \| y \|$ and since P, Q are arbitrary in $P_0(H)^n$ and y is arbitrary in $H_{11} \cap H_1^{\perp}$, we get $\| \gamma^{-1} y \| \leq \frac{1}{m} \| y \| \;\forall y \in H_{11} \cap H_1^{\perp}$. Thus γ^{-1} is bounded. Also $\| \gamma^{-1} \| \leq \frac{1}{m}$.

Conversely, suppose that γ has a bounded inverse on $P_0(H)^n$. Since $H^n \neq 0$, we have $\|\gamma^{-1}\| \neq 0$ and therefore $\| \gamma^{-1} \| > 0$. Since $\gamma : P_0(H)^n \to B(H)^n$ is bijective, each $y \in H_{11} \cap H_1^{\perp}$ is $\gamma(x)$ for a unique $x \in H^n$. So, the relation $\| \gamma^{-1} \| \leq \| \gamma^{-1} \| \| y \| \;\forall y \in H_{11} \cap H_1^{\perp}$ can be written as $\| \gamma^{-1} \gamma(x)) \| \geq \| \gamma^{-1} \| \| x \| \;\forall y \in H_{11} \cap H_1^{\perp}$.

Which shows that $\delta_{P,Q}(X)$ is bounded from below.

Corollary 3.1.8: Given $P, Q \in P_0(H)$ then $\delta_{P,Q}$ is continuous.

Proof: First we assume that $P \perp Q$. For an arbitrary $x \in H^n$, $\| x \| = 1$, then $x = Px + Qx$ and $\| x \|^2 = \| Px \|^2 + \| Qx \|^2 \geq \| Px \|^2$ and $\| x \|^2 = \| Px \|^2 + \| Qx \|^2 \geq \| Qx \|^2$. Thus both Px and Qx are bounded by 1 and so is PX and QX. Suppose that $\delta_{P,Q}$ is continuous at 0, then we can get some $\lambda > 0$ such that for all $y \in H$ with $\| y \| < \lambda$ then $\| \gamma y \| < 1$. Now for $x \in H$ and $x \neq 0$ then $\lambda \left(\frac{x}{2\|x\|}\right) = \frac{\lambda}{2}$, so $\| \gamma(\lambda \frac{x}{\|x\|}) \| < 1$. By the linearity of PX −XQ and homogeneity of the norm, we get
$1 \geq \| \gamma(\lambda \frac{x}{\|x\|}) \| = \| \lambda \frac{\gamma}{2\|x\|} \| = \frac{\lambda}{2\|x\|} \| \gamma x \|$ and therefore $\| (PX - XQ)x \| \leq M \| x \|$ with $M = \frac{2}{\lambda}$.

In the following discussion, we consider the positivity of the operator $\delta_{P,Q}$ on H^n.

Lemma 3.1.9: The product of two commuting positive operators $P \in P_0(H^n)$ and $X \in B(H)$ on H^n is also positive on H^n.

Proof: Let $P \neq 0$ and define a sequence of operators $\{S_{n=1}^{\infty}\}$ by $S_1 = P \| P \|$, $S_n + 1 = S_n - S_n^2 = S_n(I - S_n)$ for polynomials S_n in P and hence selfadjoint operators that commute with P for all $n \in N$. So that $P = \| P \| \sum_{n=1}^{\infty} \sum S_n^2$. For every $x \in H^n$, $\sum_{n=1}^{\infty} \sum \| S_n x \|^2 < \infty$ so that $\| S_n x \| \to 0$. Now, $\langle PXx, x \rangle = \| P \| \sum_{n=1}^{\infty} \langle XS_n x, S_n x \rangle \geq 0$

Lemma 3.2.0: Let H^n be a finite dimensional Hilbert space and $P, Q \in P_0(H^n)$ such that $ranP \subseteq ranQ$ and $X \in B(H)$ a positive operator on H^n that commutes with both P and Q. Then $\| PX - XQ \|^2 \geq \| PX \|^2 - \| XQ \|^2$ and $\| PX \|^2 \leq \| XQ \|^2$

Proof: We invoke vector majorization thus: Given that $P, Q \in P_0(H)$ then $P - Q$ is an orthogonal projection onto $H_{11} \cap H_1^{\perp}$ along $(H_{11} \cap H_1^{\perp})^{\perp}$ and $ranP,\ ranQ \in H_{11} \cap H_1^{\perp}$. Let $p \in ranP,\ p = \{p_i\}_{i=1}^{m}$ and $q \in ranQ,\ q = \{q_i\}_{i=1}^{n}$. For suitable bases, we can obtain the matrices for P, Q and $X \in B(H)$ such that the Hilbert-Schmidt norm applies as follows; $\| P \|_2 = (\sum_{i=1}^{m} \sum_{j=1}^{n} | p_{ij} |^2)^{\frac{1}{2}}$, $\| Q \|_2 = (\sum_{i=1}^{m} \sum_{j=1}^{n} | q_{ij} |^2)^{\frac{1}{2}}$ and $\| X \|_2 = (\sum_{i=1}^{m} \sum_{j=1}^{n} | x_{ij} |^2)^{\frac{1}{2}}$. Then $\sum_i^k q[i] \leq \sum_i^k p[i]$ and for an arbitrary $\sum_i^k x[i]$.

$$\| PX - XQ \|_2 = (\sum_{i=1}^{m} \sum_{j=1}^{n} | p_{ij}x_{ij} - x_{ij}q_{ij} |^2)^{\frac{1}{2}}$$

$$\geq (\sum_{i=1}^{m} \sum_{j=1}^{n} | p_{ij}x_{ij} |^2)^{\frac{1}{2}} - (\sum_{i=1}^{m} \sum_{j=1}^{n} | q_{ij}x_{ij} |^2)^{\frac{1}{2}}$$

$$= (\sum_{i=1}^{m} \sum_{j=1}^{n} | p_{ij}x_{ij} |^2)^{\frac{1}{2}} - (\sum_{i=1}^{m} \sum_{j=1}^{n} | x_{ij}q_{ij} |^2)^{\frac{1}{2}}$$

$$= \| PX \|_2 - \| XQ \|_2 \leq 0$$

Theorem 3.2.1: Let $P, Q \in P_0(H)$ such that $P \geq Q$ and an arbitrary positive operator $X \in B(H)$. Then $\delta_{P,Q}(X) = PX - XQ$ is positive.

Proof: It suffices to show that $\delta_{P,Q}(X)$ has square roots. If $P = Q$ then $\delta_{P,Q}(X) = 0$ then $\delta_{P,Q}(X) \geq 0$, non-negative. Suppose that $P - Q \neq 0$ then $0 \leq (P - Q) \leq I \Rightarrow 0 \leq (PX - XQ) \leq I$ for an arbitrary positive operator $X \in B(H)$.
Now $\| (PX - XQ) \| - (PX - XQ)$ must then satisfy the condition that $0 \leq \| (PX - XQ) \| - (PX - XQ) \leq I$ and so we can find an operator $\check{A}$ such that $\widetilde{A^2} = \| (PX - XQ) \|^{-} (PX - XQ)$
Then $S = (\sqrt{\| (PX - XQ) \|})\tilde{A}$ satisfies $S^2 = (PX - XQ)$. We then set $Z = I - (PX - XQ)$ and $V = I - S$. The operator V should have the property $(I - V)^2 = I - Z$, that is implicitly expressed as

$$V = \frac{1}{2}(Z + V^2) \tag{4.2.1}$$

Now $0 \leq Z \leq I$, and V chosen is such that $0 \leq V \leq I$.

Conversely, if $0 \leq V \leq I$ and satisfy equation (4.2.1) above, then $S = I - V$ is a positive square root of T. We apply method of successive approximations to solve (4.2.1). We set $V_0 = I$ and define V_n recursively by

$$V_{n+1} = \frac{1}{2}(Z + V_n^2), \quad n = 0, 1, 2, \qquad (4.2.2).$$

We show that V_n converges strongly to a solution of equation (4.2.1).

$$\text{Let } 0 \leq V_n \leq I \tag{4.2.3}$$

This is obviously true for a positive integer n.

$$\langle V_{n+1} x, x \rangle = \frac{1}{2}\langle Zx, x \rangle + \frac{1}{2} \| V_n x \|^2 \quad \forall\, x \in H^n \tag{4.2.4}$$

which implies that $V_{n+1} \geq 0$

Now $V_0 < 1$.

Suppose that $V_n \leq I$,

then equation (4.2.4) gives

$\langle V_{n+1} x, x\rangle \leq \frac{1}{2}\langle Ix, x\rangle + \frac{1}{2} \| x \|^2 = \langle Ix, x\rangle$ (since $Z \leq I$) and $V_n \leq I$. Thus $V_{n+1} \leq I$.

Consequently, $\| V_n \| \leq I \ \forall \ n \in \mathbb{N}$. Now we show that $V_n \leq V_{n+1} \forall n \in \mathbb{N}, \ V \backslash \{0\}$ i.e., $V_{n+1} - V_n \geq 0$. Next, we observe that V_n is a polynomial in Z with non-negative coefficients. Now this is true for $n = 0$ (for $V_1 - V_0 = \frac{1}{2}(Z - I)$.
We observe that

$$V_{n+1} - V_n = \frac{1}{2}(Z + V_n^2) - \frac{1}{2}(Z - V_{n-1}^2) \tag{4.2.5}$$

(It is noted that V_{n-1} and V_n are both polynomials in Z and so $V_n \leftrightarrow V_{n-1}$). Suppose that $V_n - V_{n-1}$ is a polynomial in Z with non-negative coefficients, then equation (4.2.5) shows that $V_{n+1} - V_n$ is also a polynomial in Z with non-negative coefficients for each non-negative n. Next, we show that

$$Z^k \geq 0 \tag{4.2.6}$$

For $k = 0, 1, 2, \ldots$ If $k = 2j$, then $\langle Z^k x, x\rangle = \| Z^j x \|^2 \geq 0, \ \forall \ x \in H^n$. Using equation (4.2.6) and the fact that each $V_{n+1} - V_n$ is a polynomial in Z with non-negative coefficients, we see that $V_{n+1} - V_n \geq 0$ for all the non-negative integer n.

The sequence (V_n) satisfies,

$$0 \leq V_n \leq V_{n+1} \leq I, \ n = 0, 1, 2, \tag{4.2.7}$$

and so, there is a self-adjoint operator $V \in B(H)$ such that

$$V_n \leftrightarrow V, \ V_n \leq V \leq I, \ n = 0, 1, 2 \tag{4.2.8}$$

By equation (4.2.8), we see that the operator V is a solution of equation (4.2.4). Letting $n \to \infty$ we have from equation (4.2.2)

$$V = S - \lim V_{n+1}$$

$$= S - lim \frac{1}{2}(Z + V_n^2)$$

$$= \frac{1}{2}(Z + V_n)$$

then $S = I - V$ is a square root of $(PX - XQ)$.

4. Conclusion

We have shown that $PX - XQ$ is bounded, continuous everywhere and positive i.e., $\| PX - XQ \| \geq 0$ for positive operators P, Q and an arbitrary operator X. For objective two, we have approximated the norm of $\delta_{P,Q}$ by the formula $\| \delta_{P,Q} \| = \{\sum | \alpha |^2\}^{\frac{1}{2}} - \{\sum | \beta |^2\}^{\frac{1}{2}}$ and that this norm is bounded.

5. References

1. Anderson JH, Foias C. Properties which normal operators share with normal derivation and related operators. Pacific J. Math. 1976; 61:313-325.
2. Cabello JC, Peralta AM. Weak-2-local symmetric maps on C^*-algebras. Linear Algebra Appl. 2016; 494:32-43.
3. Jorda E, Peralta AM. Stability of derivations under weak 2-local continuous pertubations. Aequationes Mathematicae, 2016.
4. Curto RE. The Spectra of elementary operators. Indiana University Mathematics Journal, Indiana, 1983.
5. Johnson BE. Continuity of centralizers on Banach algebras. Amer. J. Math. 1969; 91:1-10.

6. Kadison RV, Lance EC, Ringrose JR. Derivations and Automorphisms of operator algebras II, J. of Functional Analysis. 1947; 1:204-221.
7. Kadison RV. Local derivations, J. d'lgebra. 1990; 130:494-509.
8. Kaplansky IM. Modules over operators algebras. Amer. J. Math. 1953; 75:839-858.
9. Matej B. Characterizations of Derivations on Some Normed Algebras with Involution. Journal of Algebra. 1992; 152:454-462.
10. Niazi M, Peralta AM. Weak-2-local derivations on $\mathbb{M}_n$, Jstor. 2017; 31(6):1687-1708.
11. Ptak V. Derivations, commutators and radicals. Manuscripta. Math. 1978; 23:355-362.
12. Ringrose JR. Automatic continuity of derivations of operator algebras J. London Math. Soc. 1972; 5(2):432.
13. Salah M. Generalized derivations and C^*-algebras An. St. Univ. Ovidius Constanta. 2009; 17(2):123-130.
14. Salah M. Some recent results on operator commutators and related operators with applications.
15. Sakai, Shoichiro. On conjecture of Kaplansky Tohoku Math J. 1960; 12(2):31-33.
16. Semrl P. Local automorphisms and derivations on $B(H)$. Proc. Amer. Math Soc. 1997; 125:2677-2680.
17. Sinclair AM. Continuity of derivations and a problem of Kaplansky, Amer. J. Math. 1968; 90:1067-1073.
18. Sommerset DWB. The proximinality of the centre of a C^*-algebra, J. Approx. Theory. 1997; 89:114, 117.
19. Yanai H, Takeuchi K, Takane Y. Projection matrices, generalized inverse matrices and singular value decomposition, Springer, New York, 2011.