Int. j. adv. multidisc. res. stud. 2022; 2(2):347-349

Received: 17-01-2022
Accepted: 27-02-2022

International Journal of Advanced Multidisciplinary Research and Studies

ISSN: 2583-049X

A cusp form example as an ETA product

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Abstract
Modular forms are important in several branches of mathematics especially in number theory and they have interesting properties. For example, they have lots of internal symmetries and they are finite vector spaces over complex field. Dedekind-Eta function is one of the most important examples for half integral weight of modular forms. In this short note, we give an example for cusp form which is subspace of modular forms by using product of certain Dedekind-Eta functions. Moreover, some codes are given that using in Magma Computational Algebra System. This work is a part of master thesis of the first and the second author.

Keywords: Modular Form, Cusp Form, ETA Product, Dedekind-ETA Functions

Introduction

Modular forms are holomorphic functions on the upper half plane that satisfy fundamental symmetry and growth conditions. Roughly speaking, one can think about them as functions that have certain transformation behaviour under Mobius transformations [5].

Modular forms are used in physics as well as mathematics. It is related to the string theory. In the second author's master thesis, we consider connection of modular forms with string theory via moduli spaces of Riemann surfaces.

Definition 1: Let $SL(2,\mathbb{Z})$ be the modular group, namely,

$$\Gamma = SL(2,\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z}, ad - bc = 1 \right\}$$

Let *f* be a complex-valued function on the upper half-plane

$$\mathcal{H} := \{ z \in \mathbb{C} : Im(z) > 0 \}$$

satisfying the following conditions:

1. *f* is holomorphic on $\mathcal{H}$,
2. for any $z \in \mathcal{H}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

   where *k* is a positive integer.

3. *f* is holomorphic at ∞.

Then *f* is called a *modular form of weight k* for the modular group Γ.

A modular form *f* that vanishes at ∞ is called a *cusp form*.

A standart reference for the subject is certainly [6].

Space of modular forms is denoted by $M_k(\Gamma)$ and cusp forms by $S_k(\Gamma)$.

Let *k* be an integer. Then *k* + 1/2 is defined a *half-integer*. One can define modular forms of half-integral weight. In first author's master thesis, we review half-integral weight modular forms, and we give some examples.

Theorem 2: [5] $M_k(\Gamma)$ and $S_k(\Gamma)$ are finite dimensional vector spaces over $\mathbb{C}$.
One can define modular forms for the subgroups of Γ, for instance for the most famous subgroup *congruence subgroup*, namely,

$$\Gamma_0(N) \coloneqq \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \ (mod\ N) \}.$$

In this case, we add to the definition "level *N* ". It is clear that a modular form for Γ is of level 1.

Definition 3: Let $z \in \mathcal{H}$. Then *the Dedekind-eta function* is defined by the product

$$\eta(z) = e^{2\pi iz/24} \prod_{n=1}^{\infty}(1 - e^{2\pi inz}).$$

Theorem 4: Dimension of the space $S_6(\Gamma_0(3))$ is 1.
Proof: Using MAGMA [3], we can create this space with the following code:

```
>C:=CuspForms(3,6);
```

Then we can ask for the dimension of C:

```
>Dimension(C);
```

And we have the answer 1.

Of course, one can calculate this dimension by the formula in chapter 3 of [4].

Since $S_k(\Gamma)$ finite dimensional vector space over $\mathbb{C}$, it is natural to ask for its basis.

Theorem 5: Basis of the space $S_6(\Gamma_0(3))$ is

```
q - 6*q^2 + 9*q^3 + 4*q^4 + 6*q^5 - 54*q^6 - 40*q^7 + 168*q^8 + 81*q^9 - 36*q^10
    - 564*q^11 + 36*q^12 + 638*q^13 + 240*q^14 + 54*q^15 - 1136*q^16 + 882*q^17
    - 486*q^18 - 556*q^19 + 24*q^20 - 360*q^21 + 3384*q^22 - 840*q^23 +
    1512*q^24 - 3089*q^25 - 3828*q^26 + 729*q^27 - 160*q^28 + 4638*q^29 -
    324*q^30 + 4400*q^31 + 1440*q^32 - 5076*q^33 - 5292*q^34 - 240*q^35 +
    324*q^36 - 2410*q^37 + 3336*q^38 + 5742*q^39 + 1008*q^40 - 6870*q^41 +
    2160*q^42 + 9644*q^43 - 2256*q^44 + 486*q^45 + 5040*q^46 - 18672*q^47 -
    10224*q^48 - 15207*q^49 + O(q^50).
```

Proof: Again, using MAGMA, we can compute the basis with the following code:

```
>E := Basis(CuspForms (3, 6), 20) [1];

>E;
```

This concludes the proof.
Now, we are ready to give the main theorem.

Main Theorem: Assume the setting above. Then $(\eta(z)\,\eta(3z))^6 \in S_6(\Gamma_0(3))$
Proof: Firstly, let us compute this eta product in MAGMA:

```
>C<i> := ComplexField();
```

```
>R<z> := PowerSeriesRing(C);

>e<q> := DedekindEta(z);

>a := e^24;

>b := Evaluate(a, q^3);

>c := (a*b)^(1/4);
```

Hence, we have

```
q - 6*q^2 + 9*q^3 + 4*q^4 + 6*q^5 - 54*q^6 - 40*q^7 + 168*q^8 + 81*q^9 - 36*q^10 - 564*q^11 + 36*q^12 + 638*q^13 + 240*q^14+ 54*q^15 - 1136*q^16 + 882*q^17 - 486*q^18 - 556*q^19 + 24*q^20 - 360*q^21 + O(q^22)
```

Since $S_6(\Gamma_0(3))$ has dimension 1, this eta product is automatically in this space. One can check this with the following code:

```
>E := Basis(CuspForms(3, 6),20) [1];

>C<i> := ComplexField();

>R<z> := PowerSeriesRing(C);

>e<q> := DedekindEta(z);

>a := e^24;

>b := Evaluate(a, q^3);

>c := (a*b)^(1/4);

>c-E;
```

and by running this code in MAGMA, we get

```
O($.1^20)
```

which concludes the proof, i.e., $(\eta(z)\,\eta(3z))^6 \in S_6(\Gamma_0(3))$.

Remark 6: Proof of the main theorem can also be obtained by Proposition 25, Proposition 26 and Proposition 27 of [5]. Here, we want to show how Magma could be used in computations of modular forms.

Conclusion

Modular forms have computational aspects. On the other hand, computation on the eta products are easy. Therefore, it is important to know which form of eta products that modular forms have. There are recent papers in the literature as in [1] and [2]. Hence this topic has still attracted some attention.

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