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Generalised Frobenius Partitions with Colours and Repetitions

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Abstract
In this paper we study the partition function $c\phi_{k,h}(n)$, the number of generalised Frobenius partitions of n with k colours and h repetitions. This provides a common generalisation of the two functions $c\phi_{l,k}(n)$, and $c\phi_{k,l}(n)$ of George E. Andrews. In particular, we develop a method leading to representations of all the generating functions of $c\phi_{k,h}(n)$ as sums of infinite products.
We also obtain the Hardy – Ramanujan – Rademacher series for $c\phi_{k,h}(n)$ on the lines of L. W. Kolitsch. The existence of such series for $c\phi_{1,k}(n)$ and $c\phi_{k,l}(n)$ was asked for by Andrews and later obtained by Kolitsch.
Finally, we extend the results on q-binomial coefficients and q-series representation of Andrews to our function $c\phi_{k,h}(n)$. Andrews has established the two congruences $c\phi_{1,2}(5n+3) \equiv c\phi_{2,1}(5n+3) \equiv O \pmod 5$. We show that the analogous congruence $c\phi_{2,2}(5n+3) \equiv O \pmod 5$ is false for n = 2. We also study generalised Frobenius partitions with some restriction on its parts.

Keywords: Frobenius Partitions, General Principle

1. Introduction

A generalised Frobenius partition or simply an F-partition of a positive integer n is a two-rowed array of non-negative integers.

$$\begin{pmatrix} a_1 & \dots & a_r \\ b_1 & \dots & b_r \end{pmatrix}$$

Where each row is arranged in non-increasing order and $n = r + \sum_{i=1}^{r} (a_i + b_i)$. Let $c\phi_{k,h}(n)$ denote the number of those F-partitions of n in which each part is repeated at most h times and is taken from k copies of the non-negative integers which are ordered as follows : $m_i < n_j$ if m < n or if m = n and i< j, where i and j denote the copy of the non-negative integers. $c\phi_{k,h}(n)$ is called the number of F-partitions of n with k colours and h repetitions.

Let $c\phi_{k,h}(q)$ be the generating function of $c\phi_{k,h}(n)$ so that

$$\phi_{k,h}(q) = \sum_{n=O}^{\infty} c\phi_{k,h}(n)\, q^n.$$

For example, the F-partitions enumerated by $c\phi_{2,2}(1)$ are

$$\begin{pmatrix} o_1 \\ o_1 \end{pmatrix} \begin{pmatrix} o_2 \\ o_1 \end{pmatrix} \begin{pmatrix} o_1 \\ o_2 \end{pmatrix} \begin{pmatrix} o_2 \\ o_2 \end{pmatrix}$$

And those enumerated by $c\phi_{2,2}(2)$ are

$$\begin{pmatrix} 1_1 \\ o_1 \end{pmatrix} \begin{pmatrix} 1_2 \\ o_1 \end{pmatrix} \begin{pmatrix} 1_1 \\ o_2 \end{pmatrix} \begin{pmatrix} 1_2 \\ o_2 \end{pmatrix} \begin{pmatrix} o_1 \\ 1_1 \end{pmatrix} \begin{pmatrix} o_1 \\ 1_2 \end{pmatrix} \begin{pmatrix} o_2 \\ 1_1 \end{pmatrix} \begin{pmatrix} o_2 \\ 1_2 \end{pmatrix}$$

$$\begin{pmatrix} o_2 & o_1 \\ o_2 & o_1 \end{pmatrix}\begin{pmatrix} o_2 & o_2 \\ o_2 & o_2 \end{pmatrix}\begin{pmatrix} o_2 & o_2 \\ o_2 & o_1 \end{pmatrix}\begin{pmatrix} o_2 & o_2 \\ o_1 & o_1 \end{pmatrix}\begin{pmatrix} o_2 & o_1 \\ o_2 & o_2 \end{pmatrix}$$

$$\begin{pmatrix} o_2 & o_1 \\ o_1 & o_1 \end{pmatrix}\begin{pmatrix} o_1 & o_1 \\ o_2 & o_1 \end{pmatrix}\begin{pmatrix} o_1 & o_1 \\ o_1 & o_1 \end{pmatrix}\begin{pmatrix} o_1 & o_1 \\ o_2 & o_2 \end{pmatrix}$$

and

$$C\phi_{2,2}(q) = 1 + 4q + 17q^2 + \dots .$$

The F-partitions enumerated by $C\phi_{3,2}(1)$ are

$$\begin{pmatrix} o_1 \\ o_1 \end{pmatrix}\begin{pmatrix} o_1 \\ o_2 \end{pmatrix}\begin{pmatrix} o_1 \\ o_3 \end{pmatrix}\begin{pmatrix} o_2 \\ o_1 \end{pmatrix}\begin{pmatrix} o_2 \\ o_2 \end{pmatrix}\begin{pmatrix} o_2 \\ o_3 \end{pmatrix}\begin{pmatrix} o_3 \\ o_1 \end{pmatrix}\begin{pmatrix} o_3 \\ o_2 \end{pmatrix}\begin{pmatrix} o_3 \\ o_3 \end{pmatrix}$$

and those enumerated by $C\phi_{3,2}(2)$ are

$$\begin{pmatrix} 1_1 \\ o_1 \end{pmatrix}\begin{pmatrix} 1_2 \\ o_1 \end{pmatrix}\begin{pmatrix} 1_3 \\ o_1 \end{pmatrix}\begin{pmatrix} 1_1 \\ o_2 \end{pmatrix}\begin{pmatrix} 1_2 \\ o_2 \end{pmatrix}\begin{pmatrix} 1_3 \\ o_2 \end{pmatrix}\begin{pmatrix} 1_1 \\ o_3 \end{pmatrix}\begin{pmatrix} 1_2 \\ o_3 \end{pmatrix}\begin{pmatrix} 1_3 \\ o_3 \end{pmatrix}$$

$$\begin{pmatrix} 0_1 \\ 1_1 \end{pmatrix}\begin{pmatrix} 0_1 \\ 1_2 \end{pmatrix}\begin{pmatrix} 0_1 \\ 1_3 \end{pmatrix}\begin{pmatrix} 0_2 \\ 1_1 \end{pmatrix}\begin{pmatrix} 0_2 \\ 1_2 \end{pmatrix}\begin{pmatrix} 0_2 \\ 1_3 \end{pmatrix}\begin{pmatrix} 0_3 \\ 1_1 \end{pmatrix}\begin{pmatrix} 0_3 \\ 1_2 \end{pmatrix}\begin{pmatrix} 0_3 \\ 1_3 \end{pmatrix}$$

$$\begin{pmatrix} o_1 & o_1 \\ o_1 & o_1 \end{pmatrix}\begin{pmatrix} o_2 & o_2 \\ o_2 & o_2 \end{pmatrix}\begin{pmatrix} o_3 & o_3 \\ o_3 & o_3 \end{pmatrix}\begin{pmatrix} o_1 & o_1 \\ o_2 & o_2 \end{pmatrix}\begin{pmatrix} o_1 & o_1 \\ o_3 & o_3 \end{pmatrix}\begin{pmatrix} o_2 & o_2 \\ o_1 & o_1 \end{pmatrix}$$

$$\begin{pmatrix} o_2 & o_2 \\ o_3 & o_3 \end{pmatrix}\begin{pmatrix} o_3 & o_3 \\ o_1 & o_1 \end{pmatrix}\begin{pmatrix} o_3 & o_3 \\ o_2 & o_2 \end{pmatrix}\begin{pmatrix} o_1 & o_1 \\ o_3 & o_2 \end{pmatrix}\begin{pmatrix} o_1 & o_1 \\ o_2 & o_1 \end{pmatrix}\begin{pmatrix} o_1 & o_1 \\ o_3 & o_1 \end{pmatrix}$$

$$\begin{pmatrix} o_3 & o_2 \\ o_1 & o_1 \end{pmatrix}\begin{pmatrix} o_2 & o_1 \\ o_1 & o_1 \end{pmatrix}\begin{pmatrix} o_3 & o_1 \\ o_1 & o_1 \end{pmatrix}\begin{pmatrix} o_2 & o_2 \\ o_3 & o_2 \end{pmatrix}\begin{pmatrix} o_2 & o_2 \\ o_2 & o_1 \end{pmatrix}\begin{pmatrix} o_2 & o_2 \\ o_3 & o_1 \end{pmatrix}$$

$$\begin{pmatrix} o_3 & o_2 \\ o_2 & o_2 \end{pmatrix}\begin{pmatrix} o_2 & o_1 \\ o_2 & o_2 \end{pmatrix}\begin{pmatrix} o_3 & o_1 \\ o_2 & o_2 \end{pmatrix}\begin{pmatrix} o_3 & o_3 \\ o_3 & o_2 \end{pmatrix}\begin{pmatrix} o_3 & o_3 \\ o_2 & o_1 \end{pmatrix}\begin{pmatrix} o_3 & o_3 \\ o_3 & o_1 \end{pmatrix}$$

$$\begin{pmatrix} o_3 & o_2 \\ o_3 & o_3 \end{pmatrix}\begin{pmatrix} o_2 & o_1 \\ o_3 & o_3 \end{pmatrix}\begin{pmatrix} o_3 & o_1 \\ o_3 & o_3 \end{pmatrix}\begin{pmatrix} o_3 & o_2 \\ o_3 & o_2 \end{pmatrix}\begin{pmatrix} o_3 & o_2 \\ o_2 & o_1 \end{pmatrix}\begin{pmatrix} o_3 & o_2 \\ o_3 & o_1 \end{pmatrix}$$

$$\begin{pmatrix} o_2 & o_1 \\ o_3 & o_2 \end{pmatrix}\begin{pmatrix} o_3 & o_1 \\ o_3 & o_2 \end{pmatrix}\begin{pmatrix} o_2 & o_1 \\ o_2 & o_1 \end{pmatrix}\begin{pmatrix} o_2 & o_1 \\ o_3 & o_1 \end{pmatrix}\begin{pmatrix} o_3 & o_1 \\ o_2 & o_1 \end{pmatrix}\begin{pmatrix} o_3 & o_1 \\ o_3 & o_1 \end{pmatrix}$$

And

$$C\phi_3{,}2(q) = 1 + 9q + 54q^2 + \dots .$$

George E. Andrews (2) has studied extensively the two functions $c\phi_{1,k}(n) = \phi_k(n)$ and $c\phi_{k,1}(n) = c\phi_k(n)$. The former function enumerates the F-partitions of n in which the parts repeat at most k times and the latter enumerates those F-partitions of n in which the parts are distinct and are coloured withy k given colours. Andrews (2) has obtained infinite product representations for $C\phi_{1,1}(q) = \phi_1 (q)$, $c\phi_{1,2}(q) = \phi_2(q)$, $C\phi_{1,3}(q) = \phi_3(q)$, $c\phi_{2,1}(q) = C\phi_2(q)$ and has expressed $C\phi_{3,1}(q) = C\phi_3(q)$, $c\phi_{2,1}(q) = c\phi_2(q)$as a sum of two infinite products. In this paper we obtain representations of $c\phi_{k,h}(q)$ as sums of infinite products for k = 2, 3 h = 2, 3 and k = 1, 2, h = 4.

2. Preliminary Results: In this paper we prove some results which are needed to obtain representations of $C\phi_{k,h}(q)$ as sums of infinite products for k = 2, 3, h = 2, 3 and k = 1, 2, h = 4. The results obtained based on a particular case of the following results of Andrews (3, Lemma 3) :

(2.1) $(z\alpha q)_\infty (z\beta q)_\infty (z^{-1}\alpha^{-1})_\infty (z^{-1}\beta^{-1})_\infty$

$$= A_o(\alpha,\beta, q) \sum_{n=-\infty}^{\infty} q^{n^2+n} \alpha^n\beta^n z^{2n}$$

$$- \beta^{-1}A_o (\alpha q, \beta, q) \sum_{n=-\infty}^{\infty} q^{n^2} \alpha^n\beta^n z^{2n-1},$$

Where z, α, β are non-zero, $|q| < 1$ and

(2.2) $A_o (\alpha, \beta, q) = (-q)_\infty (q)_\infty^{-1} (-\alpha\beta^{-1} q;q^2)_\infty (-\alpha^{-1}\beta q; q^2)_\infty$.

Choosing $\alpha = \omega$, $\beta = \omega^2$ in (2.1), where $\omega = \exp (2\pi i/3)$ and observing that

$$\prod_{n=1}^{\infty}(1-q^{2n-1}+q^{4n-2}) = (-q^3; q^6)_\infty / (-q ; q^2)_\infty$$

we obtain :

(2.3) $\prod_{n=o}^{\infty}(1+zq^{n-1}+z^2q^{2n+2}) (1+z^{-1} q^n+z^{-2} q^{2n})$

$$= A_o (q) \sum_{n=-\infty}^{\infty} q^{n^2+n} z^{2n}$$

$$= B_o (q) \sum_{n=-\infty}^{\infty} q^{n^2} z^{2n-1,}$$

Where $A_o(q)$ and $B_o (q)$ are defined by

(2.4) $A_o(q) = \dfrac{(-q^2;q^2)_\infty(-q^3;q^6)_\infty}{(q)_\infty}$

(2.5) $B_o(q) = \dfrac{(-q;q^2)_\infty(-q^6;q^6)_\infty}{(q)_\infty}$

Remark 1: Equating the constant terms in (2.3) we obtain Andrews' representation for $C\phi1,2$ (q) [2, eq (5.11)] .
For studying various properties of ϕ_k (n) and $c\phi_k(n)$, Andrews (2) makes use of the following General Principle. We state it here for convenience, since our discussions are also based on it.

General Principle : If $f_A (z,q) = f_A(z) = \Sigma P_A (m,n) z^m q^n$ denotes the generating function for P_A (m,n), the number of ordinary partitions of n into m parts subject to the set of restrictions A, then $f_A (zq) f_B(z^{-1})$, has as its constant term the generating function.

$$\phi_{A, B}(q) = \sum_{n=O}^{\infty} \phi_{A, B}(n) q^n,$$

Where $\phi_{A, B}(n)$ is the number of F-partitions

$$\begin{pmatrix} a_1 & \dots & a_r \\ b_1 & \dots & b_r \end{pmatrix}$$

in which the top row is subject to the set of restrictions A and the bottom row is subject to the set of restrictions B.
From the above General Principle it is clear that $C\phi_{k,h}(q)$ is the constant term in

$$\prod_{n=o}^{\infty}(1+zq^{n-1}+ \dots + z^hq^{hn+h})^k (1+z^{-1} q^n+ \dots + z^{-h} q^{hn})^k.$$

Lemma 1: For a > O, b, c integers and |q| < 1,

(2.6) $\sum_{n_1,n_2=-\infty}^{\infty} q^{a(n_1^2+n_2^2+n_1n_2)+bn_1+cn_2}$

$= (q^{6a}; q^{6a})_\infty (-q^{3a+2b-c};q^{6a})_\infty (-q^{3a-2b+c}; q^{6a})_\infty$

$\quad X (q^{2a}; q^{2a})_\infty (-q^{a+c}; q^{2a})_\infty (-q^{a-c}; q^{2a})_\infty$

$+ q^{a+b} (q^{6a}; q^{6a})_\infty (-q^{6a+2b-c};q^{6a})_\infty (-q^{-2b+c}; q^{6a})_\infty$

$\quad X (q^{2a}; q^{2a})_\infty (-q^{2a+c}; q^{2a})_\infty (-q^{-c}; q^{2a})_\infty$

Proof : Consider

$$\sum_{n_1,n_2=-\infty}^{\infty} q^{a(n_1^2+n_2^2+n_1n_2)+bn_1+cn_2}$$

$$= \sum_{n_1=-\infty}^{\infty} q^{4an_1^2+2bn_1} \sum_{n_2=-\infty}^{\infty} q^{a(n_2^2+2n_1 n_2)+cn_2}$$

$$+ \sum_{n_1=-\infty}^{\infty} q^{a(4n_1^2+4n_1+1)+2bn_1+b} \sum_{n_2=-\infty}^{\infty} q^{a(n_2^2+2n_1n_2)+an_2+cn_2}$$

(after grouping separately terms with n_1 even and n_1 odd).

$$= \sum_{n_1=-\infty}^{\infty} q^{3an_1^2+(2b-c)n_1} \sum_{n2=-\infty}^{\infty} q^{a(n_2+n_1)^2+c(n_2+n_1)}$$

$$+ q^{a+b} \sum_{n_1=-\infty}^{\infty} q^{3an_1^2+(3a+2b-c)n_1} \sum_{n2=-\infty}^{\infty} q^{a(n_2+n_1)^2+(a+c)(n_2+n_1)}$$

$$= \sum_{n_1=-\infty}^{\infty} q^{3an_1^2+(2b-c)n_1} \sum_{n2=-\infty}^{\infty} q^{an_2^2+cn_2}$$

$$+ q^{a+b} \sum_{n_1=-\infty}^{\infty} q^{3an_1^2+(3a+2b-c)n_1} \sum_{n2=-\infty}^{\infty} q^{an_2^2+(a+c)n_2}$$

(on changing n_2 to $n_2 - n_1$).

If we now use the triple product identity of Jacobi

(2.7) $\sum_{n=-\infty}^{\infty} q^{n^2} z^n = (q^2;q^2)_\infty (-qz;q^2)_\infty (-qz^{-1};q^2)_\infty$

for z ≠ O and |q| < 1 in the above summations we obtain (2.6) and this proves Lemma 1.
****

3. Representations of $C\phi_{2.2}(q)$ and $C\phi_{3,2}$ (q).

Theorem 1. For |q| < 1,

(3.1) $C\phi_{2,2}(q) = \dfrac{(-q^2;q^2)_\infty^2 (-q^3;q^6)_\infty^2 (-q^2;q^4)_\infty^2}{(q)_\infty^2}$

$+ 2q \dfrac{(-q;q^2)_\infty^2 (-q6;q^6)_\infty^2 (q^4;q^4)_\infty (-q^4;q^4)_\infty^2}{(q)_\infty^2}$

Proof : From the General Principle we know that $C\phi 2.2(q)$ is the constant term in

$$\prod_{n=o}^{\infty}(1+zq^{n+1}+z^2q^{2n+2})^2\,(1+z^{-1}q^n+z^{-2}q^{2n})^2.$$

Squaring (2.2.3) and equating the constant terms, we get

$$(3.2)\ C\phi_{2,2}\,(q) = A_o\,(q)^2 \sum_{n=-\infty}^{\infty} q^{2n^2} + B_o(q)^2 \sum_{n=-\infty}^{\infty} q^{2n^2-2n+1}$$

If we now use Jacobi's triple product identity for the two summations in (3.2), we get (3.1) and this proves Theorem 1.

Theorem 2: For $|q| < 1$.

$$(3.3)\ C\phi_{3,2}(q) = \frac{(-q^2;q^2)^3_\infty(-q^3;q^6)^3_\infty(q^{12};q^{12})_\infty(q^4;q^4)_\infty}{(q)^3_\infty}$$

$$X\,[(-q^6;q^{12})^2_\infty(-q^2;q^4)^2_\infty+4q^2(-q^{12};q^{12})^2_\infty(-q^4;q^4)^2_\infty\,]$$

$$+\ 6q\,\frac{(-q^2;q^2)^2_\infty(-q^3;q^6)^2_\infty(q;q^2)^2_\infty(-q^6;q^6)^2_\infty\,(q^4;q^4)_\infty(q^{12};q^{12})_\infty}{(q)^3_\infty}$$

$$X\,[(-q^6;q^{12})^2_\infty(-q^4;q^4)^2_\infty+q(-q^{12};q^{12})^2_\infty(-q^2;q^4)^2_\infty\,]$$

Proof: We know that $C\phi_{3,2}(q)$ is the constant term in

$$\prod_{n=o}^{\infty}(1+zq^{n+1}+z^2q^{2n+2})^3\,(1+z^{-1}q^n+z^{-2}q^{2n})^3.$$

Cubing (2.3) and equating the constant terms, we get

$$(3.4)\ C\phi_{3,2}\,(q) = A_o\,(q)^3 \sum_{n_1,n_2=-\infty}^{\infty} q^{2(n_1^2+n_2^2+n_1n_2)}$$

$$+\ 3A_o(q)B_o(q)^2 \sum_{n_1,n_2=-\infty}^{\infty} q^{2(n_1^2+n_2^2+n_1n_2)-n_1-2n_2+1}$$

Where $A_o\,(q)$ and $B_o\,(q)$ are defined by (2.4) and (2.5) respectively. If we now use Lemma 1 with a = 2, b = c = O for the first series and a = 2, b = -1, c = -2 for the second series on the right-hand side of the equation (3.4), we obtain (3.3) and this proves Theorem 2.

****

4. Representations of $C\phi_{2,3}(q)$ and $C\phi_{3,3}(q)$.

To obtain the representations of $C\phi_{2,3}(q)$ and $C\phi_{3,3}(q)$ we first extend Andrews' lemma (3, Lemma 3).

Lemma 2. For z, α, β, γ, all non-zero and $|q| < 1$,

$$(4.1)\ \prod_{n=1}^{\infty}(1-z\alpha q^n)\,(1-z\beta q^n)\,(1-z\gamma q^n)$$

$$X\ (1-z^{-1}\alpha^{-1}q^{n-1})\,(1-z^{-1}\beta^{-1}q^{n-1})\,(1-z^{-1}\gamma^{-1}q^{n-1})$$

$$= A_o\,(\alpha,\beta,q)\,B\,(\alpha,\beta,\gamma,q) \sum_{n=-\infty}^{\infty} (-1)^{3n}\alpha^n\beta^n\gamma^n q^{1/2(3n^2+3n)}z^{3n}$$

$$+\ q\,A_o\,(\alpha,\beta,q)\,B\,(\alpha,\beta,\gamma q,q)$$

$$X \sum_{n=-\infty}^{\infty} (-1)^{3n}\alpha^n\beta^n\gamma^{n+1}Q^{(\frac{1}{2}3n2+5n)}z^{3n+1}$$

$$+\ q^3\,A_o\,(\alpha,\beta,q)\,B\,(\alpha,\beta,\gamma q^2,q)$$

X $\sum_{n=-\infty}^{\infty}$ $(-1)^{3n}\alpha^n\beta^n\gamma^{n+2}q^{1/2}(3n^2+7n)z^{3n+2}$

$+ q\beta^{-1} A_o (\alpha q, \beta, q) B (\alpha, \beta, \gamma q^{3/2}, q)$

X $\sum_{n=-\infty}^{\infty}$ $(-1)^{3n}\alpha^n\beta^n\gamma^{n+1}q^{1/2}(3n^2+3n)z^{3n}$

$- q^3\beta^{-1} A_o (\alpha q, \beta, q) B (\alpha, \beta, \gamma q^{5/2}, q)$

X $\sum_{n=-\infty}^{\infty}$ $(-1)^{3n}\alpha^n\beta^n\gamma^{n+2}q^{1/2}(3n^2+5n)z^{3n+1}$

$- q^2\beta^{-1} A_o (\alpha q, \beta, q) B (\alpha, \beta, \gamma q^{1/2}, q)$

X $\sum_{n=-\infty}^{\infty}$ $(-1)^{3n}(\alpha\beta\gamma)^{n+1}q^{1/2}(3n^2+7n)z^{3n+2}$

Where $A_o (\alpha, \beta, q)$ is as defined in (2.2) and

(4.2) $B(\alpha, \beta, \gamma, q) = (q^6; q^6) (-\alpha\beta\gamma^2Q^3;q^6)_\infty$

X $(-\alpha^{-1}\beta^{-1}\gamma^2q^3;q^6)_\infty(q)_\infty^{-1}$.

Proof : By Andrews' lemma (identity (2.1) and Jacobi's triple product identity, we find
(4.3) $(1-z\alpha q^n) (1-z\beta q^n) (1-z\gamma q^n)$

X $(1-z^{-1}\alpha^{-1}q^{n-1}) (1-z^{-1}\beta^{-1}q^{n-1}) (1-z^{-1}\gamma^{-1}q^{n-1})$

$= \frac{1}{(q)_\infty} A_o (\alpha, \beta, q) \sum_{N=-\infty}^{\infty} q^{N^2+N} \alpha^N\beta^N z^{2N} \sum_{n=-\infty}^{\infty} (-1)^n z^n\gamma^n q\frac{(n+1)}{2}$

$= \frac{\beta^{-1}}{(q)_\infty} A_o (\alpha q, \beta, q) \sum_{N=-\infty}^{\infty} q^{N^2+N} \alpha^N\beta^N z^{2N-1} \sum_{n=-\infty}^{\infty} (-1)^n z^n\gamma^n q\frac{(n+1)}{2}$

Consider

$$\sum_{N=-\infty}^{\infty} q^{N^2+N} \alpha^N\beta^N z^{2N} \sum_{n=-\infty}^{\infty} (-1)^n z^n\gamma^n q\frac{(n+1)}{2}$$

$$= \sum_{N=-\infty}^{\infty} q^{N^2+N} \alpha^N\beta^N z^{2N} \sum_{n=-\infty}^{\infty} (-1)^{n-2N} z^{n-2N}\gamma^{n-2N} q\frac{(n-2N+1)}{2}$$

$$= \sum_{n=-\infty}^{\infty}(-1)^n z^n\gamma^n q^{1/2}(n^2+n) \sum_{N=-\infty}^{\infty}\alpha^N\beta^N\gamma^{-2N}q^{3N^2}-2nN$$

$$= \sum_{n=-\infty}^{\infty}(-1)^{3n} z^{3n}\gamma^{3n} q^{1/2}(9n^2+3n) \sum_{N=-\infty}^{\infty}\alpha^N\beta^N\gamma^{-2N}q^{3N^2}-6nN$$

$$+ \sum_{n=-\infty}^{\infty}(-1)^{3n+1} z^{3n+1}\gamma^{3n+1} q^{1/2}[(3n+1)^2+(3n+1)]$$

$$\sum_{N=-\infty}^{\infty}\alpha^{N}\beta^{N}\gamma^{-2N}q^{3N^{2}}-2N(3n+1)$$

$$+\sum_{n=-\infty}^{\infty}(-1)^{3n+2}z^{3n+2}\gamma^{3n+2}q^{1/2}[\ ^{(3n+2)^{2}+(3n+2)}]$$

$$\mathrm{X}\sum_{N=-\infty}^{\infty}\alpha^{N}\beta^{N}\gamma^{-2N}q^{3N^{2}-2N(3n+2)}$$

(after grouping separately terms with n ≡ 0, 1, 2 (mod 3)).

$$=\sum_{n=-\infty}^{\infty}(-1)^{3n}z^{3n}\alpha^{n}\beta^{n}\gamma^{n}q^{1/2}\,(3n^{2}+3n)$$

$$\mathbf{X}\sum_{N=-\infty}^{\infty}\alpha^{N-n}\beta^{N-n}\gamma^{-2(N-n)}q^{3}\,(N-n)^{2}$$

$$+\,q\sum_{n=-\infty}^{\infty}(-1)^{3n+1}z^{3n+1}\alpha^{n}\beta^{n}\gamma^{n+1}q^{1/2}\,(3n^{2}+5n)$$

$$\mathbf{X}\sum_{N=-\infty}^{\infty}\alpha^{N-n}\beta^{N-n}\gamma^{-2(N-n)}q^{3}\,(N-n)^{2}-2(N-n)$$

$$+\,q^{3}\sum_{n=-\infty}^{\infty}(-1)^{3n}z^{3n+2}\alpha^{n}\beta^{n}\gamma^{n+1}q^{1/2}\,(3n^{2}+7n)$$

$$\mathbf{X}\sum_{N=-\infty}^{\infty}\alpha^{N-n}\beta^{N-n}\gamma^{-2(N-n)}q^{3}\,(N-n)^{2}-4(N-n)$$

$$=(q^{6};q^{6})_{\infty}(-\alpha\beta\gamma^{-2}q^{3};q^{6})_{\infty}\,(-\alpha^{-1}\beta^{-1}\gamma^{2}q^{3};q^{6})_{\infty}$$

$$\mathrm{X}\sum_{n=-\infty}^{\infty}(-1)^{3n}\alpha^{n}\beta^{n}\gamma^{n+1}q^{1/2}\,(3n^{2}+3n)z^{3n}$$

$$+q\,(q^{6};q^{6})_{\infty}(-\alpha\beta\gamma^{-2}q;q^{6})_{\infty}\,(-\alpha^{-1}\beta^{-1}\gamma^{2}q^{5};q^{6})_{\infty}$$

$$\mathrm{X}\sum_{n=-\infty}^{\infty}(-1)^{3n+1}\alpha^{n}\beta^{n}\gamma^{n+1}q^{1/2}\,(3n^{2}+5n)z^{3n+1}$$

$$+\,q^{3}\,(q^{6};q^{6})_{\infty}(-\alpha\beta\gamma^{-2}q^{-1};q^{6})_{\infty}\,(-\alpha^{-1}\beta^{-1}\gamma^{2}q^{7};q^{6})_{\infty}$$

$$\mathrm{X}\sum_{n=-\infty}^{\infty}(-1)^{3n}\alpha^{n}\beta^{n}\gamma^{n+2}q^{1/2\,(3n^{2}+7n)}$$

(on changing N to N + n and using (2.7)).

Lemma 2 now follows if we observe that

$$\sum_{N=-\infty}^{\infty}q^{N^{2}}\ \alpha^{N}\beta^{N}z^{2N-1}\sum_{n=-\infty}^{\infty}(-1)^{n}z^{n}\gamma^{n}q\frac{(n+1)}{2}$$

$$=z^{-1}\sum_{N=-\infty}^{\infty}q^{N^{2}}+^{N}(\alpha/q)^{N}\beta^{N}z^{2N}\sum_{n=-\infty}^{\infty}(-1)^{n}z^{n}\gamma^{n}q\,\frac{(n+1)}{2}$$

Corollary 1. For z non – zero and |q| < 1.

(4.4) $\prod_{n=O}^{\infty}$ $(1+zq^{n+1}+z^2q^{2n+2}+z^3q^{3n+3})\ (1+z^{-1}q^n+z^{-2}q^{2n}+z^{-3}q^{3n})$

$$= \frac{(q^2;q^2)_\infty (q;q^2)^2_\infty (q^6;q^6)_\infty}{(q)^3_\infty}$$

X $[(-q^3;q^6)^2_\infty \sum_{n=-\infty}^{\infty} q^{1/2(3n^2+3n)}z^{3n}$.

$+ q\,(-q;q^6)_\infty(-q^5;q^6)_\infty \sum_{n=-\infty}^{\infty} q^{1/2(3n^2+5n)}z^{3n+1}$

$+ q^3\,(-q^{-1};q^6)_\infty(-q^7;q^6)_\infty \sum_{n=-\infty}^{\infty} q^{1/2(3n^2+7n)}z^{3n+2}]$.

Proof: We have

$$A_o(i,-i,q) = \frac{(-q)_\infty (q;q^2)_\infty}{(q)_\infty}$$

$$= \frac{(q^2;q^2)_\infty (q;q^2)^2_\infty}{(q)^2_\infty}$$

$$iA_o(iq,-i,q) = \frac{i(-q)_\infty}{(q)_\infty} \prod_{n=1}^{\infty} = (1-q^{2n})(1-q^{2n-2}) = O$$

And

$$B(i,-i,-1,q) = \frac{(q^6;q^6)_\infty (-q^3;q^6)^2_\infty}{(q)_\infty}$$

Substituting these values in Lemma 2, we obtain (4.4).

Remark 2: Equating the constant terms in (4.4), we obtain Andrews representation for $C\phi_{1,3}(q) = \phi_3(q)$.
Theorem 3: For |q| < 1.

$$(4.5)\ C\phi_{2,3}(q) = \frac{(q^2;q^2)^2_\infty (q;q^2)^4_\infty (q^6;q^6)^3_\infty}{(q)^6_\infty}$$

X $[(-q^3;q^6)^6_\infty + 2q^2(-q;q^6)^2_\infty\ (-q^5;q^6)^2_\infty (-q^{-1};q^6)_\infty(-q^7;q^6)_\infty]$.

Proof : From the General Principle it is clear that $C\phi_{2,3}(q)$ is the constant term in

$\prod_{n=O}^{\infty}$ $(1+zq^{n+1}+z^2q^{2n+2}+z^3q^{3n+3})^2\ (1+z^{-1}q^n+z^{-2}q^{2n}+z^{-3}q^{3n})^2$

Squaring (4.4) and extracting the coefficients of z^0, we find

$$(4.6)\ C\phi_{2,3}(q) = \frac{(q^2;q^2)^2_\infty (q;q^2)^4_\infty (q^6;q^6)^2_\infty}{(q)^6_\infty}$$

$$\times \left[(-q^3;q^6)_\infty^4 \sum_{n=-\infty}^{\infty} q^{3n^2}\right.$$

$$\left. + 2q^4 (-q;q^6)_\infty(-q^5;q^6)_\infty (-q^{-1}; q^6)_\infty(-q^7;q^6)_\infty \sum_{n=-\infty}^{\infty} q^{3n^2+2n-2} \right]$$

Now using Jacobi's triple product identity for the two summations on the right hand side of (4.6), we obtain (4.5)

Theorem 4. For $|q| < 1$.

$$(4.7)\ C\phi_{3,3}(q) = \frac{(q^2;q^2)3(q;q^2)_\infty^6 (q^6;q^6)_\infty^4 (q^{18};q^{18})_\infty}{(q)_\infty^9}$$

$$\times\ (-q^3;q^6)_\infty^6 (-q^9;q^{18})_\infty^2 (-q^3;q^6)_\infty^2 + 4q^3 (-q^{18};q^{18})_\infty^2 (-q^6;q^6)_\infty]$$

$$+ q^2 (-q;q^6)_\infty^3 (-q^5;q^6)_\infty^3 (-q^{12};q^{18})_\infty (-q^6;q^{18})_\infty (-q^6;q^6)_\infty^2]$$

$$+ q^6 (-q^{21};q^{18})_\infty(-q^{-3}; q^{18})_\infty(-q^9;q^6)_\infty(-q^{-3};q^6)_\infty$$

$$+ q^8 (-q^{-1};q^6)_\infty^3 (-q^7;q^6)_\infty^3 (-q^{15};q^{18})_\infty (-q^3;q^{18})_\infty (-q^9;q^6)_\infty$$

$$\times (-q^{-3};q^6)_\infty + q^9(-q^{24}; q^{18})_\infty(-q^{-6};q^{18})_\infty(-q^{-12};q^6)_\infty(-q^{-6};q^6)_\infty]$$

$$+ 6q^2 (-q3;q^6)_\infty^2 (-q;q^6)_\infty (-q^5;q^6)_\infty (-q^{-1};q^6)_\infty (-q^7;q^6)_\infty$$

$$\times [(-q^9;q^{18})_\infty^2 (-q^5; q^6)_\infty(-q;q^6)_\infty$$

$$+ 2q^4(-q^{18};q^{18})_\infty^2 (-q^8;q^6)_\infty(-q^{-2};q^6)_\infty]$$

Proof: We know that $C\phi_{3,3}$ (q) is the constant term in

$$\prod_{n=0}^{\infty} (1+zq^{n+1}+z^2q^{2n+2}+z^3q^{3n+3})^3 (1+z^{-1}q^n+z^{-2}q^{2n}+z^{-3}q^{3n})^3.$$

Cubing (4.4) and equating the constant terms, we find

$$(4.8)\ C\phi_{3,3}(q) = \frac{(q^6;q^6)_\infty^3 (q^2;q^2)_\infty^3 (q;q^2)_\infty^6}{(q)_\infty^9}$$

$$\times \left[(-q^3;q^6)_\infty^6 \sum_{m,n=-\infty}^{\infty} q^{3(m^2+n^2+mn)}\right.$$

$$+ q^2(-q;q^6)_\infty^3 (-q^5;q^6)_\infty^3 \sum_{m,n=-\infty}^{\infty} q^{3(m^2+n^2+mn)+3m+3n}$$

$$+ q^8 (-q^{-1};q^6)_\infty^3 (-q^7;q^6)_\infty^3 \sum_{m,n=-\infty}^{\infty} q^{3(m^2+n^2+mn)+6m+6n}$$

$$+ 6q^2 (-q^3;q^6)_\infty^2 (-q;q^6)_\infty(-q^5;q^6)_\infty (-q^{-1};q^6)_\infty (-q^7;q^6)_\infty$$

$$\times \sum_{m,n=-\infty}^{\infty} q^{3(m^2+n^2+mn)+2m+n}$$

If we now use Lemma 1 for each of the four series on the right-hand side of (4.8), we obtain (4.7).
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5. Representations of $C\phi_{1,4}(q)$ and $C\phi_{2,4}(q)$.

To obtain representations of Cϕ1,4(q) and Cϕ2,4(q) we further extend Andrews' lemma (3, Lemma 3),. For this we first prove a lemma.

Lemma 3. Let z, α, β, γ, ∂ be all non-zero and let |q| < 1. If

(5.1) S (α, β, γ, ∂, z, q)

$$= \frac{1}{(q)_\infty} \sum_{m=-\infty}^{\infty} = (-1)^{3m} z^{3m} \alpha^m \beta^m \gamma q^{1/2\,(3m^2+3m)}$$

$$\text{X} \sum_{n=-\infty}^{\infty} (-1)^n z^n \partial^n q^{\binom{n+1}{2}}]$$

Then

(5.2) S (α, β, γ, ∂, z, q)

$$= C(\alpha,\beta,\gamma,\partial,q) \sum_{n=-\infty}^{\infty} (\alpha\beta\gamma\partial)^{2n} q^{8n^2+4n} z^{8n}$$

$$-\partial q\, C(\alpha,\beta,\gamma,\partial q^{-1},q) \sum_{n=-\infty}^{\infty} (\alpha\beta\gamma\partial)^{2n} q^{8n^2+6n} z^{8n+1}$$

$$+\partial^2 q^3\, C(\alpha,\beta,\gamma,\partial q^2,q) \sum_{n=-\infty}^{\infty} (\alpha\beta\gamma\partial)^{2n} q^{8n^2+8n} z^{8n+2}$$

$$-\partial^3 q^6\, C(\alpha,\beta,\gamma,\partial q^3,q) \sum_{n=-\infty}^{\infty} (\alpha\beta\gamma\partial)^{2n} q^{8n^2+10n} z^{8n+3}$$

$$+\partial^4 q^{10}\, C(\alpha,\beta,\gamma,\partial q^4,q) \sum_{n=-\infty}^{\infty} (\alpha\beta\gamma\partial)^{2n} q^{8n^2+12n} z^{8n+4}$$

$$-\partial^4 q^{15}\, C(\alpha,\beta,\gamma,\partial q^5,q) \sum_{n=-\infty}^{\infty} (\alpha\beta\gamma\partial)^{2n} q^{8n^2+14n} z^{8n+5}$$

$$+\partial^6 q^{21}\, C(\alpha,\beta,\gamma,\partial q^6,q) \sum_{n=-\infty}^{\infty} (\alpha\beta\gamma\partial)^{2n} q^{8n^2+16n} z^{8n+6}$$

$$-\partial^7 q^{28}\, C(\alpha,\beta,\gamma,\partial q^7,q) \sum_{n=-\infty}^{\infty} (\alpha\beta\gamma\partial)^{2n} q^{8n^2+18n} z^{8n+7}$$

Where

(5.3) $C(\alpha,\beta,\gamma,\partial,q) = (q^{12};q^{12})_\infty\,(-\alpha,\beta\gamma\partial^{-3}q^6q^{12})_\infty$

$$\text{X}\ (-\alpha^{-1}\beta^{-1}\gamma^{-1}\partial^3 q^6;q^{12})_\infty\,(q)_\infty^{-1}$$

Proof : Consider

(5.4) S (α, β, γ, ∂, z, q) $(q)_\infty$

$$= \sum_{m=-\infty}^{\infty} (-1)^{3m} z^{3m} \alpha^m \beta^m \gamma^m q^{1/2\,(3m2+3m)} \ \text{x} \sum_{n=-\infty}^{\infty} (-1)^n 2^n \partial^n Q\left(\frac{n+1}{2}\right)$$

$$= \sum_{m=-\infty}^{\infty} (-1)^{3m} z^{3m} \alpha^m \beta^m \gamma^m q^{1/2(3m2+3m)}$$

$$\text{x} \sum_{n=-\infty}^{\infty} (-1)^{n-3m} z^{n-3m} \partial^{n-3m} q^{\frac{(n-3m+1)}{2}}]$$

(on changing n to n – 3m).

$$= \sum_{n=-\infty}^{\infty} (-1)^n z^n \partial^n q^{1/2n(n+1)} \sum_{m=-\infty}^{\infty} (\alpha\beta\gamma\partial^{-3})^m q^{6m^2-3mn}$$

$$= \sum_{n=-\infty}^{\infty} z^{8n} \partial^{8n} q^{8n^2+4n} \sum_{m=-\infty}^{\infty} (\alpha\beta\gamma\partial^{-3})^m q^{6(m-2n)^2}$$

$$- q \sum_{n=-\infty}^{\infty} z^{8n+1} \partial^{8n+1} q^{8n2+6n}$$

$$\text{x} \sum_{m=-\infty}^{\infty} = (\alpha\beta\gamma\partial^{-3})^m q^{6(m-2n)^2-3(m-2n)}$$

$$+ q^3 \sum_{n=-\infty}^{\infty} z^{8n+2} \partial^{8n+2} q^{8n2+8n}$$

$$\text{x} \sum_{m=-\infty}^{\infty} (\alpha\beta\gamma\partial^{-3})^m q^{6(m-2n)^2-6(m-2n)}$$

$$- q^6 \sum_{n=-\infty}^{\infty} z^{8n+2} \partial^{8n+3} q^{8n^2+10n}$$

$$\text{x} \sum_{m=-\infty}^{\infty} (\alpha\beta\gamma\partial^{-3})^m q^{6(m-2n)^2-9(m-2n)}$$

$$+ q^{10} \sum_{n=-\infty}^{\infty} z^{8n+4} \partial^{8n+4} q^{8n2+12n}$$

$$\text{x} \sum_{m=-\infty}^{\infty} (\alpha\beta\gamma\partial^{-3})^m q^{6(m-2n)^2-12(m-2n)}$$

$$- q^{15} \sum_{n=-\infty}^{\infty} z^{8n+5} \partial^{8n+5} q^{8n2+14n}$$

$$\text{x} \sum_{m=-\infty}^{\infty} (\alpha\beta\gamma\partial^{-3})^m q^{6(m-2n)^2-15(m-2n)}$$

$$+ q^{21} \sum_{n=-\infty}^{\infty} z^{8n+6} \partial^{8n+6} q^{8n2+16n}$$

$$\text{X} \sum_{m=-\infty}^{\infty} (\alpha\beta\gamma\partial^{-3})^m q^{6(m-2n)^2-18(m-2n)}$$

$$- q^{28} \sum_{n=-\infty}^{\infty} z^{8n+8} \partial^{7n+7} q^{8n2+18n}$$

$$X \sum_{m=-\infty}^{\infty} (\alpha\beta\gamma\partial^{-3})^m q^{6(m-2n)^2-21(m-2n)}$$

(by grouping separately terms with n ≡ r (mod 8), where r = 0, 1, ..., 7).
If we now change m to m + 2n and use Jacobi's triple product identity in we obtain (5.2). This proves Lemma 3.

Lemma 4. For z, α, β, γ, ∂ all non – zero |q| < 1.

(5.5) $\prod_{n=1}^{\infty} (1-z\alpha q^n)(1-z\beta q^n)(1-z\gamma q^n)(1-z\partial q^n)$.

$X (1-z^{-1}\alpha^{-1}q^{n-1})(1-z^{-1}\beta^{-1}q^{n-1})(1-z^{-1}\gamma^{-1}q^{n-1})(1-z^{-1}\partial^{-1}q^{n-1})$

$= S(\alpha, \beta, \gamma, \partial, z, q)$

$x [A_o(\alpha, \beta, q) B(\alpha, \beta, \gamma, q) + q\beta^{-1}\gamma A_o(\alpha q, \beta, q) B(\alpha, \beta, \gamma q^{3/2,} q)]$

$= S(\alpha, \beta, \gamma, \partial, z, q)$

$x [q\gamma z A_o(\alpha, \beta, q) B(\alpha, \beta, \gamma q, q) - q^3\beta^{-1}\gamma^2 z A_o(\alpha q, \beta, q) B(\alpha, \beta, \gamma q^{5/2,} q)]$

$+ S(\alpha, \beta, \gamma, \partial, z, q) x [qyzA_0(\alpha, \beta, Q) B(\alpha, \beta, \gamma Q, Q) - Q^3\beta^{-1}\gamma^2 Z A_0(\alpha Q, \beta, Q) B(\alpha, \beta, \gamma Q^{5/2}, Q)] + S(\alpha Q^2, \beta, \gamma, \partial, Z, Q)$

$x [q^3\gamma^2 z^2 A_o(\alpha, \beta, q) B(\alpha, \beta, \gamma q^2, q) - q^2\alpha\gamma z^2 z A_o(\alpha q, \beta, q) B(\alpha, \beta, \gamma q^{1/2,} q)]$

Proof : Using Lemma 2 and Jacobi's triple product identity, we find

(5.6) $\prod_{n=1}^{\infty} (1-z\alpha q^n)(1-z\beta q^n)(1-z\gamma q^n)(1-z\partial q^n)$

$X (1-z^{-1}\alpha^{-1}q^{n-1})(1-z^{-1}\beta^{-1}q^{n-1})(1-z^{-1}\gamma^{-1}q^{n-1})(1-z^{-1}\partial^{-1}q^{n-1})$

$$= \frac{1}{(q)_\infty} [A_o(\alpha, \beta, q) B(\alpha, \beta, \gamma, q)$$

$$x \sum_{m=-\infty}^{\infty} (-1)^{3m}\alpha^m\beta^m\gamma^m q^{1/2^{(3m^2+3m)}} z^{3m} \sum_{n=-\infty}^{\infty} (-1)^n z^n \partial^n q^{\frac{(n+1)}{2}}]$$

$$+ \frac{1}{(q)_\infty} [qA_o(\alpha, \beta, q) B(\alpha, \beta, \gamma q, q)$$

$$x \sum_{m=-\infty}^{\infty} (-1)^{3m}\alpha^m\beta^m\gamma^{m+1} q^{1/2^{(3m^2+3m)}} z^{3m+1} \sum_{n=-\infty}^{\infty} (-1)^n z^n \partial^n q^{\frac{(n+1)}{2}}]$$

$$+ \frac{1}{(q)_\infty} [q^3A_o(\alpha, \beta, q) B(\alpha, \beta, \gamma q^2, q)$$

$$X \sum_{m=-\infty}^{\infty} (-1)^{3m}\alpha^m\beta^m\gamma^m q^{1/2^{(3m^2+7m)}} z^{3m+2} \sum_{n=-\infty}^{\infty} (-1)^n z^n \partial^n q^{\frac{(n+1)}{2}}]$$

$$+ \frac{1}{(q)_\infty} [q\beta^{-1}A_o(\alpha q, \beta, q) B(\alpha, \beta, \gamma q^{3/2}, q)$$

$$X \sum_{m=-\infty}^{\infty} (-1)^{3m}\alpha^m\beta^m\gamma^{m+1} q^{1/2^{(3m^2+3m)}} z^{3m} \sum_{n=-\infty}^{\infty} (-1)^n z^n \partial^n q^{\frac{(n+1)}{2}}]$$

$+ \frac{1}{(q)_\infty}$ $[q^3\beta^{-1}A_o(\alpha q, \beta, q)\, B(\alpha, \beta, \gamma q^{5/2}, q)$

$$X \sum_{m=-\infty}^{\infty} (-1)^{3m}\alpha^m\beta^m\gamma^{m+2} q^{1/2(3m^2+5m)} z^{3m+1} \sum_{n=-\infty}^{\infty} (-1)^n z^n \partial^n q^{\binom{n+1}{2}}]$$

$- \frac{1}{(q)_\infty}$ $[q^2\beta^{-1}A_o(\alpha q, \beta, q)\, B(\alpha, \beta, \gamma q^{1/2}, q)$

$$X \sum_{m=-\infty}^{\infty} (-1)^{3m}(\alpha\beta\gamma)^{m+1} q^{1/2(3m^2+7m)} z^{3m+2} \sum_{n=-\infty}^{\infty} (-1)^n z^n \partial^n q^{\binom{n+1}{2}}]$$

Lemma 4 now follows if we observe that the first series product on the right-hand side of (5.6) is S $(\alpha, \beta, \gamma, \partial, z, q)$, the second is S $(\alpha q, \beta, \gamma, \partial, z, q)$ and so on.
Using Lemma 3 and 4 we obtain the Leurent expansion of the product (5.5) which we state it in the following lemma.

Lemma 5. For z, α, β, γ, ∂ all non-zero and $|q| < 1$.

(5.7) $\prod_{n=1}^{\infty} (1-z\alpha q^n)(1-z\beta q^n)(1-z\gamma q^n)(1-z\partial q^n)$

$X\, (1-z^{-1}\alpha^{-1}q^{n-1})(1-z^{-1}\beta^{-1}q^{n-1})(1-z^{-1}\gamma^{-1}q^{n-1})(1-z^{-1}\partial^{-1}q^{n-1})$

$$= A_1 \sum_{n=-\infty}^{\infty} (\alpha,\beta,\gamma,\partial)^{2n} q^{8n^2+4n} z^{8n}$$

$$+ A_2 \sum_{n=-\infty}^{\infty} (\alpha,\beta,\gamma,\partial)^{2n} q^{8n^2+6n} z^{8n+1}$$

$$+ A_3 \sum_{n=-\infty}^{\infty} (\alpha,\beta,\gamma,\partial)^{2n} q^{8n^2+8n} z^{8n+2}$$

$$+ A_4 \sum_{n=-\infty}^{\infty} (\alpha,\beta,\gamma,\partial)^{2n} q^{8n^2+10n} z^{8n+3}$$

$$+ A_5 \sum_{n=-\infty}^{\infty} (\alpha,\beta,\gamma,\partial)^{2n} q^{8n^2+12n} z^{8n+4}$$

$$+ A_6 \sum_{n=-\infty}^{\infty} (\alpha,\beta,\gamma,\partial)^{2n} q^{8n^2+14n} z^{8n+5}$$

$$+ A_7 \sum_{n=-\infty}^{\infty} (\alpha,\beta,\gamma,\partial)^{2n} q^{8n^2+16n} z^{8n+6}$$

$$+ A_8 \sum_{n=-\infty}^{\infty} (\alpha,\beta,\gamma,\partial)^{2n} q^{8n^2+18n} z^{8n+7}$$

Where

$A_1 = E_1C(\alpha,\beta,\gamma,\partial,q) - \alpha^{-2}\beta^{-2}\gamma^{-2}\partial^5 q^{16} E_2C(\alpha q,\beta,\gamma,\partial q^7,q)$
$\quad + \alpha^{-2}\beta^{-2}\gamma^{-2}\partial^4 q^9 E_3C(\alpha q^2,\beta,\gamma,\partial q^6,q)$

$A_2 = -\partial q E_1C(\alpha,\beta,\gamma,\partial q^{-1},q) + E_2C(\alpha q,\beta,\gamma,\partial,q)$

$\quad -\alpha^{-2}\beta^{-2}\gamma^{-2}\partial^5 q^{14} E_3C(\alpha q^2,\beta,\gamma,\partial q^6,q)$

$A_3 = \partial^2 q^3 E_1 C(\alpha,\beta,\gamma,\partial q^2,q) - \partial q E_2 C(\alpha q,\beta,\gamma,\partial q^{-1},q)$

$+ E_3 C(\alpha q^2,\beta,\gamma,\partial,q)$

$A_4 = -\partial^3 q^6 E_1 C(\alpha,\beta,\gamma,\partial q^3,q) - \partial^2 q^3 E_2 C(\alpha q,\beta,\gamma,\partial q^2,q)$

$-\partial q E_3 C(\alpha q^2,\beta,\gamma,\partial,q^{-1},q)$

$A_5 = \partial^4 q^{10} E_1 C(\alpha,\beta,\gamma,\partial q^4,q) - \partial^3 q^6 E_2 C(\alpha q,\beta,\gamma,\partial q^3,q)$

$+ \partial^2 q^3 E_3 C(\alpha q,\beta,\gamma,\partial q^2,q)$

$A_6 = -\partial^5 q^{15} E_1 C(\alpha,\beta,\gamma,\partial q^5,q) + \partial^4 q^{10} E_2 C(\alpha q,\beta,\gamma,\partial q^4,q)$

$-\partial^3 q^6 E_3 C(\alpha q^2,\beta,\gamma,\partial q^3,q)$

$A_7 = \partial^6 q^{21} E_1 C(\alpha,\beta,\gamma,\partial q^6,q) - \partial^5 q^{15} E_2 C(\alpha q,\beta,\gamma,\partial q^5,q)$

$+ \partial^4 q^{10} E_3 C(\alpha q^2,\beta,\gamma,\partial q^4,q)$

$A_8 = -\partial^7 q^{28} E_1 C(\alpha,\beta,\gamma,\partial q^7,q) - \partial^6 q^{21} E_2 C(\alpha q,\beta,\gamma,\partial q^6,q)$

$-\partial^5 q^{15} E_3 C(\alpha q^2,\beta,\gamma,\partial q^5,q)$

And

$E_1 = A_o(\alpha,\beta,q)\,B(\alpha,\beta,\gamma,q) + q\beta^{-1}\gamma A_o(\alpha q,\beta,q)\,B(\alpha,\beta,\gamma q^{3/2},q)$

$E_2 = q\gamma A_o(\alpha,\beta,q)\,B(\alpha,\beta,\gamma q,q) - q^3\beta^{-1}\gamma^2 A_o(\alpha q,\beta,q)\,B(\alpha,\beta,\gamma q^{5/2},q)$

$E_3 = q^3\gamma^2 A_o(\alpha,\beta,q)\,B(\alpha,\beta,\gamma q^2,q) - q^2\alpha\gamma A_o(\alpha q,\beta,q)\,B(\alpha,\beta,\gamma q^{1/2},q)$

The method adopted in obtaining representations of $C\phi_{2,2}(q)$, $C\phi_{3,2}(q)$, $C\phi_{2,3}(q)$ and $C\phi_{3,3}(q)$ can now be used to get the expressions (stated in the following theorem) of $C\phi_{1,4}(q)$ and $C\phi_{2,4}(q)$ as sums of infinite products.

Theorem 5. For $|q| < 1$.

(5.7) $C\phi_{1,4}(q) = \dfrac{(-q)_\infty (q^6;q^6)_\infty (q^{12};q^{12})_\infty}{(q)^3_\infty}$

$\times (-\omega^4 q^6;q^{12})_\infty (-\omega q^6;q^{12})_\infty$

$[(-\omega^4 q;q^2)_\infty (-\omega q;q^2)_\infty (-\omega^2 q^3;q^6)_\infty (-\omega^3 q^3;q^6)_\infty$

$+ \omega q (\omega^4 q^2;q^2)_\infty (-\omega;q^2)_\infty (-\omega^2;q^6)_\infty (-\omega^3 q^6;q^6)_\infty]$

$- \omega^3 q^{16} (-\omega^2 q^7 q^{12})_\infty (-\omega^3 q^5;q^{12})_\infty$

$\times [\omega^3 q (-\omega^4 q;q^2)_\infty (-\omega q;q^2)_\infty (-\omega^2 q;q^6)_\infty (-\omega^3 q^5;q^6)_\infty]$

$- \omega^4 q^3 (-\omega^4 q^2;q^2)_\infty (-\omega;q^2)_\infty (-\omega^2;q^{-2};q^6)_\infty (-\omega^3 q^8;q^6)_\infty]$

$+ \omega^4 q^9 (-\omega^2 q^8;q^{12})_\infty (-\omega^3 q^4;q^{12})_\infty$

$x\, [\omega q^3 (-\omega^4 q;q^2)_\infty (-\omega q;q^2)_\infty (-\omega^2 q^{-1};q^6)_\infty (-\omega^3 q^7;q^6)_\infty$

$- \omega^4 q^2 (-\omega^4 q^2;q^2)_\infty (-\omega;q^2)_\infty (-\omega^2 q^2;q^6)_\infty (-\omega^3 q^4;q^6)_\infty]$

(5.8) $C\phi_{2,4}(q) = (q^{32};q^{32})_\infty [B_1^2 (-q^{16};q^{32})^2_\infty$

$+ 2q^{-10} B_2 B_8 (-q^{20};q^{32})_\infty (-q^{12};q^{32})_\infty$

$+ 2q^{-8} B_3B_7(-q^{24};q^{32})_\infty (-q^8;q^{32})_\infty$

$+ 2q^{-6} B_4B_6(-q^{28};q^{32})_\infty (-q^4;q^{32})_\infty$

$= 2q^{-4} B_5^2 (-q^{32};q^{32})_\infty^2$],

where B1,…,B8 are the values of A1,…,A8 (defined in Lemma 5) at $\alpha = \omega$, $\beta = \omega^2$, $\gamma = \omega^3$ and $\partial = \omega^4$ with $\omega = \exp(2\pi i/5)$

6. References

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