Int. j. adv. multidisc. res. stud. 2024; 4(2):111-117

Received: 16-01-2024
Accepted: 26-02-2024

International Journal of Advanced Multidisciplinary Research and Studies

ISSN: 2583-049X

About some applications of matrices

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Abstract
One of the important subjects discussed in linear algebra is the matrix, its properties and operations. Matrices have many applications in scientific fields, in different branches of mathematics and in real life. In this article, I present some applications of matrices in some of the above fields so that students are more interested in learning this area of knowledge.

Keywords: Matrix, Diagonalization

1. Introduction
This article is divided into two parts. First, recall some knowledge about matrices. In the second part, the article will introduce some applications of matrices in practical life and in some other fields.

2. Content
2.1 Some knowledge about matrices
2.1.1 Diagonal the square matrix n
Definition 1.2: A square matrix A of order n is called diagonalizable if there exists an invertible matrix P such n that $P^{-1}AP$ it is diagonal.

Theorem 1.1: **A** diagonal n square matrix A is diagonalizable if and only if A there are enough n linearly independent eigenvectors. Furthermore, the P diagonalized matrix A is a matrix whose columns are n linearly independent eigenvectors of A.

2.1.2 Powers of diagonalizable square matrices
Given A a square matrix of order n, if A diagonalizable then there exist two matrices D and P such that $A = P^{-1}DP$. Then,

$$A^k = P^{-1}D^kP; \forall k \in N$$

Which $D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$, where $\lambda_i; i = \overline{1; n}$ are the eigenvalues of the matrix A.

So $D^k = \begin{pmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n^k \end{pmatrix}$, from that it can be inferred A^k.

2.2 Some applications of matrices
2.2.1 Application to find the general term of a sequence
Given a series of numbers in recurring form, determining the general term sometimes causes many difficulties. Here the article presents how to find the general term of a series of numbers through knowledge of matrices.

Method:
- From the recurrence relation, determine the matrix A and the matrix X_n so that the relation is written in matrix form $X_n = AX_{n-1}$, and then deduce $X_n = A^n X_0$.
- Calculate A^n, from there we can conclude u_n.

Example: Find the general term of the sequence given by: $\begin{cases} u_0 = 1; u_1 = 1 \\ u_n = u_{n-1} + 6u_{n-2} \end{cases}$ $\forall n \geq 2$.

Prize.

Put $A = \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix}; X_n = \begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix}$

Then, $X_n = AX_{n-1} = A^n X_0$

$$P_A(\lambda) = det(A - \lambda I_2) = \begin{vmatrix} -\lambda & 1 \\ 6 & 1-\lambda \end{vmatrix} = \lambda^2 - \lambda - 6 = 0 \Leftrightarrow \left[\begin{matrix} \lambda = 3 \\ \lambda = -2 \end{matrix}\right.$$

$\Rightarrow A$ diagonalizable and $A = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{3}{5} & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}$

$$X_n = A^n X_0 = P^{-1} D^n P X_0 \Leftrightarrow \begin{bmatrix} u_n \\ u_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{3}{5} & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 3^n & 0 \\ 0 & (-1)^n 2^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

But

$$\Leftrightarrow \begin{bmatrix} u_n \\ u_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{3^{n+1} + (-1)^n 2^{n+1}}{5} \\ \frac{3^{n+2} + (-1)^{n+1} 2^{n+2}}{5} \end{bmatrix}$$

So, $u_n = \frac{3^{n+1} + (-1)^n 2^{n+1}}{5}$.

2.2.2 Application in Leontief open economic model

Suppose an economy consists of n production. It is called x_i the total demand value of the industry's product $i\left(i = \overline{1, n}\right)$, b_i the final demand value, x_{ik} and the intermediate demand value.

Then, the total demand for the industry's products i is

$$x_i = x_{i1} + x_{i2} + \ldots + x_{ik} + b_i \quad \text{(first)}$$

and the industry's share of input costs k for products i is

$$a_{ik} = \frac{x_{ik}}{x_k} (0 < a_{ik} < 1)$$

Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ be the input cost matrix (or technical coefficient matrix); $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is the aggregate demand matrix; $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ is the final spherical matrix.

Then, from equation (1), if $x_{ik} = a_{ik} x_k$ we substitute:

$$x_i = a_{i1} x_1 + a_{i2} x_2 + \ldots + a_{in} x_n \quad \left(i = \overline{1, n}\right)$$

Or expressed as a matrix: $X = AX + B$ (2)

The problem here is how to find the aggregate demand matrix ? (when the input coefficient matrix and final spherical matrix are known).

From (2), we have: $(I - A)X = B$, where I is the level unit matrix n.
If the matrix $(I - A)$is non-degenerate then $X = (I - A)^{-1}B$ (3)
+ The matrix $(I - A)$ is called the Leontief matrix.
+ The matrix $C = (I - A)^{-1} = (c_{ij})_{nxn}$ is called the total cost matrix. The meaningful j coefficient is: to produce a unit of value of the industry's final demand c_{ij}, the industry i needs to produce a quantity of products with a value of c_{ij}.

Example 1: Suppose in an economy there are two manufacturing industries: industry 1 and industry 2 with input cost matrix as:

$$A = \begin{pmatrix} 0.2 & 0.3 \\ 0.4 & 0.1 \end{pmatrix}$$

Indicate that the final value for the products of industry 1 and industry 2 is 10 and 20 billion VND, respectively. Determine the value of aggregate demand for each industry.

Prize.

Called $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ the aggregate demand matrix, where x_1 is the aggregate demand value of industry 1, x_2 is the aggregate demand value of industry 2.

By assumption, we have the final matrix $B = \begin{pmatrix} 10 \\ 20 \end{pmatrix}$

Then, we have: $X = (I - A)^{-1}B = \frac{1}{0.6}\begin{pmatrix} 0.9 & 0.3 \\ 0.4 & 0.8 \end{pmatrix}\begin{pmatrix} 10 \\ 20 \end{pmatrix} = \begin{pmatrix} 25 \\ \frac{100}{3} \end{pmatrix}$

So, the total demand value of industry 1 is x_1= 25 billion VND and the total demand value of industry 2 is x_2= $\frac{100}{3}$ billion VND.

Example 2: In the open economic model Leontief knows the input cost matrix is

$$A = \begin{pmatrix} 0.2 & m & 0.3 \\ 0.3 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.2 \end{pmatrix}$$

a) State the meaning of the element located in row 2, column 1 of the matrix A.
b) Find m that when the output of three industries is 400, 400, 300, the first economic industry's supply to the open economic industry is 130.
c) With the value m found in part b), find the total cost coefficient matrix and state the meaning of the element located in row 3, column 2 of this matrix.

Prize.

a) $a_{21} = 0.3$: means that to produce a unit of value in industry 1, industry 2 must directly provide this industry with a quantity of products with a value of 0.3.
b) Called $Y$the output value matrix of the three industries

$$Y = \begin{pmatrix} 400 \\ 400 \\ 300 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

By assumption, we have:

$$X_1 = a_{11}X_1 + a_{12}X_2 + a_{13}X_3 + b_1$$
$$\Leftrightarrow 400 = 0.2 \times 400 + 400m + 0.3 \times 300 + 130$$
$$\Leftrightarrow m = 0.25$$

c) With $m = 0.25$, we have:

$$A = \begin{pmatrix} 0.2 & 0.25 & 0.3 \\ 0.3 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.2 \end{pmatrix}$$

Then, the total cost matrix is:

$$C = (I - A)^{-1} = \begin{pmatrix} 1.751 & 0.769 & 0.849 \\ 0.743 & 1.538 & 0.663 \\ 0.716 & 0.769 & 1.711 \end{pmatrix}$$

$c_{32} = 0.769$ This means that to produce one unit of final demand value of industry 2, industry 3 needs to produce a quantity of products with a value of 0.769.

2.2.3 Application to solve systems of differential equations

Consider a system of differential equations $\frac{dX}{dt} = AX + F(t) \quad (1)$, where A is a square matrix of order n.

If a matrix A is diagonalizable, then there exists an P invertible matrix and a diagonal matrix D such that $A = P^{-1}DP$

Then, (1) $\Leftrightarrow \frac{dX}{dt} = P^{-1}DPX + F(t) \Leftrightarrow P\frac{dX}{dt} = DPX + PF(t)$ (2)

Put $Y = PX \Rightarrow Y' = P\frac{dX}{dt}$

Substituting into (2) we get: $Y' = DY + PF(t)$

Example: Solve the system of equations: $\begin{cases} x_1' = 3x_1 + x_2 + e^t \\ x_2' = 2x_1 + 2x_2 + t \end{cases}$

Put $A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}; X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; F(t) = \begin{pmatrix} e^t \\ t \end{pmatrix}$

$$P_A(\lambda) = det(A - \lambda I_2) = \begin{vmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 4 = 0 \Leftrightarrow \left[\begin{matrix} \lambda = 1 \\ \lambda = 4 \end{matrix}\right.$$

$\Rightarrow$ A diagonalizable and $A = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$

Put $Y = PX \Rightarrow Y' = P\frac{dX}{dt}$. Then, we have: $Y' = DY + PF(t)$

$$\Leftrightarrow \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} e^t \\ t \end{pmatrix} \Leftrightarrow \begin{cases} y_1' = 4y_1 + e^t + t \\ y_2' = y_1 + e^t - 2t \end{cases}$$

$$\Leftrightarrow \begin{cases} y_1(t) = -\frac{1}{3}e^t - \frac{t}{4} - \frac{1}{16} + C_1e^{4t} \\ y_2(t) = te^t + 2t + 2 + C_2e^t \end{cases}$$

The solution of the given system is:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = P^{-1}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}\begin{pmatrix} -\frac{1}{3}e^t - \frac{t}{4} - \frac{1}{16} + C_1e^{4t} \\ te^t + 2t + 2 + C_2e^t \end{pmatrix} = \begin{pmatrix} \frac{t}{3}e^t - \frac{2}{9}e^t + \frac{t}{2} + \frac{15}{24} + C_1e^{4t} + C_2e^t \\ -\frac{t}{3}e^t - \frac{1}{9}e^t - \frac{3}{4}t - \frac{33}{48} + C_1e^{4t} + C_2e^t \end{pmatrix}$$

2.2.4 Application to balance chemical equations

A chemical equation (or equation representing a chemical reaction) is an equation consisting of two sides connected by an arrow from left to right, the left side represents the substances participating in the reaction, the right side represents the reactants. Substances obtained after the reaction, all substances are written by their chemical formula and have appropriate coefficients placed before that chemical formula to ensure the law of mass conservation. Finding the coefficients given to the substances before and after that reaction is called balancing the chemical equation.

What should we do with the problem of balancing a chemical equation? There are many ways to balance chemical equations, one of which is the law of conservation of mass, which means that the mass of substances before and after the reaction remains unchanged, this means the number of moles of each element The chemistry involved in the reaction is conserved.

Sticking to this, we see that if we put the coefficients in front of the substances before and after reacting with unknowns, then for each chemical element we will obtain an equation.

For example: Balancing a chemical equation: $Al + HCl \rightarrow AlCl_3 + H_2$

We will set the variables x, y, z, t as coefficients first $Al; HCl; AlCl_3; H_2$

Now the equation after balancing will have the form:

$$xAl + yHCl \rightarrow zAlCl_3 + tH_2$$

With element Al: $x = z$
With element H: $y = 2t$
With element Cl: $y = 3z$

We have a system of equations: $\begin{cases} x + 0y - z + 0t = 0 \\ 0x + y + 0z - 2t = 0 \\ 0x + y - 3z + 0t = 0 \end{cases}$

Consider $\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 1 & -3 & 0 \end{pmatrix} \xrightarrow{d_3 \to d_3 - d_2} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & -3 & 2 \end{pmatrix} \xrightarrow{d_1 \to 3d_1 - d_2} \begin{pmatrix} 3 & 0 & 0 & -2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & -3 & 2 \end{pmatrix}$

At this point, the system of equations is rewritten: $\begin{cases} x + 0y + 0z - 2t = 0 \\ 0x + y + 0z - 2t = 0 \\ 0x + y - 3z + 0t = 0 \end{cases}$

Put $t = a \Rightarrow x = \frac{2}{3}a; y = 2a; z = \frac{2}{3}a$

Select $a = 3 \Rightarrow x = 2; y = 6; z = 2; t = 3$

We get the equation after balancing:

$$2Al + 6HCl \rightarrow 2AlCl_3 + 3H_2$$

2.2.5 Application in life and practice

The matrix is widely used in everyday life more than we think. With sports statistics, we use two matrix multiplication to calculate the total points scored after matches. Let's consider the following example in sports:

Example 1: In a group of 4 competing teams, the points scored for each position are as follows: 5 points for the team in 1st place, 3 points for 2nd place, and 1 point for the team in 2nd place. 3. Find the total score for each school? Which school won the award?

Location / School	1st	Monday	Tuesday
Nguyen Du	8	4	5
Nguyen Trai	6	3	7
Nguyen Hien	5	7	3
Nguyen Thi Minh Khai	7	5	4

Prize:

First, we write the results of the competition and the scores in matrix form. Set the matrix so that the number of columns of the resulting matrix is equal to the number of rows of the dot matrix.

Result matrix

$$R = \begin{pmatrix} 8 & 4 & 5 \\ 6 & 3 & 7 \\ 5 & 7 & 3 \\ 7 & 5 & 4 \end{pmatrix},$$

Point matrix

$$P = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}$$

Performing multiplication of two matrices we get

$$RP = \begin{pmatrix} 8 & 4 & 5 \\ 6 & 3 & 7 \\ 5 & 7 & 3 \\ 7 & 5 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 57 \\ 46 \\ 49 \\ 54 \end{pmatrix}$$

So, the scores of the corresponding schools are:
Nguyen Du: 57; Nguyen Trai: 46; Nguyen Hien: 49; Nguyen Thi Minh Khai: 54
Therefore, Nguyen Du is the winning school with 57 points.
Let's consider an example with football.

Example 2: In the 2018 AFC U-23 Championship, Vietnam is in Group D with the following results:

Group D			MP	W	D	L
1		Korea Republic U23	3	2	1	0
2		Vietnam U23	3	1	1	1
3		Australia U23	3	1	0	2
4		Syria U23	3	0	2	1

In there:

MP: number of matches (played)
W: number of wins (win)
D: number of draws (draw)
L: number of lost matches (lose)
P: score (point)

Calculate each team's points after the end of the group stage?

Solution: We know that each team gets 3 points for a win, 1 point for a draw, and 0 points for a loss. Set up the match result matrix and score matrix, multiply the two matrices to get the corresponding total score matrix for each team:

$$T = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 3 \\ 2 \end{pmatrix}$$

This is the P score calculated in the table below:

Group D			MP	W	D	L	GF	GA	GD	P
1		Korea Republic U23	3	2	1	0	5	3	2	7
2		Vietnam U23	3	1	1	1	2	2	0	4
3		Australia U23	3	1	0	2	5	5	0	3
4		Syria U23	3	0	2	1	1	3	-2	2

Consider another example in the business field, we use a matrix to calculate the total value.

Example 3: Mr. A has 3 fruit farms including apples, peaches and apricots. He sells \$22 a crate of apples, \$25 a crate of peaches and \$18 a crate of apricots.

Number of Boxes Each Type of Fruit			
Farm	Apple	Dig	Dream
first	280	170	210
2	185	240	190
3	110	90	0

Calculate the total amount he earned from each farm and the total amount Mr. A sold?

Solution: Similar to Example 2, we can calculate the revenue of each farm as the product of two matrices, the inventory matrix and the price per barrel matrix:

$$\begin{pmatrix} 280 & 170 & 210 \\ 185 & 240 & 190 \\ 110 & 90 & 0 \end{pmatrix} \begin{pmatrix} 22 \\ 25 \\ 18 \end{pmatrix} = \begin{pmatrix} 14190 \\ 13490 \\ 4670 \end{pmatrix}$$

The amount of money Mr. A earned from farm 1 is: 14,190 USD
The amount of money Mr. A earned from farm 2 is: 13,390 USD
The amount of money Mr. A earned from farm 3 is: 4,670 USD
We get the total amount Mr. A sold for 32,350 USD.

3. Conclusion

Through this article, the author has presented some knowledge about matrices and some examples of matrix applications to solve problems in practice, in other fields and in mathematics. From there, We can see some applications of matrices. Thereby, creating more interest for students in learning mathematics, helping them see more clearly the role of studying advanced mathematics in the University program.

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