Int. j. adv. multidisc. res. stud. 2022; 2(5):525-540

Received: 24-08-2022
Accepted: 04-10-2022

International Journal of Advanced Multidisciplinary Research and Studies

ISSN: 2583-049X

Group Analysis on One-Dimensional Heat Equation

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Abstract
We study a one-dimensional heat equation by Lie group analysis method. The constructed Lie point symmetries have been employed in reduction of the partial differential equation into simple ordinary differential equations and exact solutions obtained. A Soliton has been produced by use of a linear combination of time and space translation symmetries. We also compute conservation laws using multiplier approach.

Keywords: Heat Equation, Lie Group Analysis, Group-Invariant Solutions, Stationary Solutions, Symmetry Reductions, Solitons, Traveling Wave

1. Introduction
The one-dimensional heat equation [5],

$$\Delta \equiv u_t - hu_{xx} = 0, \tag{1.1}$$

where t and x represent time and spatial independent variables in the dependent variable u, has been a subject of study for nearly 200 years. The constant h is the diffusivity of the medium upon which heat travels. Equation (1.1) is a very interesting model of diffusion in a continuous medium and boasts of a very wide applicability and a considerable volume of rich mathematical theories have emanated from its study. It is important to mention that Equation (1.1) is intimately related to Burger's Equation [18].

2. Preliminaries
This section presents a prelude that is used in what comes after.

Local Lie groups. [6]
We will consider the transformations

$$T_\epsilon: \quad \bar{x}^i = \varphi^i(x^i, u^\alpha, \epsilon), \quad \bar{u}^\alpha = \psi^\alpha(x^i, u^\alpha, \epsilon), \tag{2.1}$$

in the Euclidean space R^n of $x = x^i$ independent variables and R^m of $u = u^\alpha$ dependent variables. The continuous parameter ϵ ranges from a neighbourhood $\mathcal{N}' \subset \mathcal{N} \subset \mathbb{R}$ of $\epsilon = 0$ for ϕ^i and ψ^α differentiable and analytic in the parameter ϵ.

Definition 2.1 Let $\mathcal{G}$ be a set of transformations in (2.1). Then $\mathcal{G}$ is a local Lie group if:

1) Given $T_{\epsilon_1}, T_{\epsilon_2} \in \mathcal{G}$, for $\epsilon_1, \epsilon_2 \in N' \subset N$, then

$T_{\epsilon_1} T_{\epsilon_2} = T_{\epsilon_3} \in \mathcal{G}$, $\epsilon_3 = \varphi(\epsilon_1, \epsilon_2) \in N$ (Closure).

2) There exists a unique $T_0 \in \mathcal{G}$ if and only if $\epsilon = 0$ such that $T_\epsilon T_0 = T_0 T_\epsilon = T_\epsilon$ (Identity).

3) There exists a unique $T_\epsilon{-1} \in \mathcal{G}$ for every transformation $T_\epsilon \in \mathcal{G}$,
where $\epsilon \in N' \subset N$ and $\epsilon^{-1} \in N$ such that
$T_\epsilon T_\epsilon{-1} = T_\epsilon{-1}\, T_\epsilon = T_0$ (Inverse).

Remark 2.2 The condition (i) is sufficient for associativity of $\mathcal{G}$.

Prolongations. Consider the system,

$$\Delta_\alpha\left(x^i, u^\alpha, u_{(1)}, \ldots, u_{(\pi)}\right) = \Delta_\alpha = 0, \tag{2.2}$$

where u^α are dependent variables with partial derivatives $u_{(1)} = \{u^\alpha_i\}$, $u_{(2)} = \{u^\alpha_{ij}\}$, . . . , $u_{(\pi)} = \{u^\alpha_{i1 \ldots i\pi}\}$, of the first, second, . . . , up to the πth-orders. We shall denote by

$$D_i = \frac{\partial}{\partial x^i} + u^\alpha_i \frac{\partial}{\partial u^\alpha} + u^\alpha_{ij} \frac{\partial}{\partial u^\alpha_j} + \ldots, \tag{2.3}$$

the total differentiation operator with respect to the variables x^i and δ^j_i, the Kronecker delta. Then

$$D_i(x^j) = \delta^j_i, ', \; u^\alpha_i = D_i(u^\alpha), \;\; u^\alpha_{ij} = D_j(D_i(u^\alpha)), \;\; \ldots, \tag{2.4}$$

where u^α_i defined in (2.4) are differential variables [6].

(1) Prolonged groups Let $\mathcal{G}$ given by

$$\bar{x}^i = \varphi^i(x^i, u^\alpha, \epsilon), \quad \varphi^i\Big|_{\epsilon=0} = x^i, \;\; \bar{u}^\alpha = \psi^\alpha(x^i, u^\alpha, \epsilon), \quad \psi^\alpha\Big|_{\epsilon=0} = u^\alpha, \tag{2.5}$$

Where $\Big|_{\epsilon=0}$ means evaluated on $\epsilon = 0$.

Definition 2.3 The construction of $\mathcal{G}$ in (2.5) is equivalent to the computation of infinitesimal transformations

$$\begin{aligned} \bar{x}^i &\approx x^i + \xi^i(x^i, u^\alpha)\epsilon, \quad \varphi^i\Big|_{\epsilon=0} = x^i, \\ \bar{u}^\alpha &\approx u^\alpha + \eta^\alpha(x^i, u^\alpha)\epsilon, \quad \psi^\alpha\Big|_{\epsilon=0} = u^\alpha, \end{aligned} \tag{2.6}$$

obtained from (2.1) by a Taylor series expansion of $\phi^i(x^i, u^\alpha, \epsilon)$ and $\psi^i(x^i, u^\alpha, \epsilon)$ in ϵ about $\epsilon = 0$ and keeping only the terms linear in ϵ, where

$$\xi^i(x^i, u^\alpha) = \frac{\partial \varphi^i(x^i, u^\alpha, \epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}, \quad \eta^\alpha(x^i, u^\alpha) = \frac{\partial \psi^\alpha(x^i, u^\alpha, \epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}. \tag{2.7}$$

Remark 2.4 By using the symbol of infinitesimal transformations, X, (2.6) becomes

$$\bar{x}^i \approx (1 + X)x^i, \quad \bar{u}^\alpha \approx (1 + X)u^\alpha, \tag{2.8}$$

Where

$$X = \xi^i(x^i, u^\alpha)\frac{\partial}{\partial x^i} + \eta^\alpha(x^i, u^\alpha)\frac{\partial}{\partial u^\alpha}, \tag{2.9}$$

is the generator $\mathcal{G}$ in (2.5).

Remark 2.5 The change of variables formula

$$D_i = D_i(\varphi^j)\bar{D}_j, \tag{2.10}$$

is employed to construct transformed derivatives from (2.1). The D_j is total differentiation $\bar{x}^i$. As a result

$$\bar{u}^\alpha_i = \bar{D}_i(\bar{u}^\alpha), \;\; \bar{u}^\alpha_{ij} = \bar{D}_j(\bar{u}^\alpha_i) = \bar{D}_i(\bar{u}^\alpha_j). \tag{2.11}$$

If we apply the change of variable formula given in (2.10) on $\mathcal{G}$ given by (2.5), we get

$$D_i(\psi^\alpha) = D_i(\varphi^j), \;\; \bar{D}_j(\bar{u}^\alpha) = \bar{u}^\alpha_j D_i(\varphi^j). \tag{2.12}$$

If we expand (2.12), we obtain

$$\left(\frac{\partial \varphi^j}{\partial x^i} + u_i^\beta \frac{\partial \varphi^j}{\partial u^\beta}\right) \bar{u}_j^\beta = \frac{\partial \psi^\alpha}{\partial x^i} + u_i^\beta \frac{\partial \psi^\alpha}{\partial u^\beta}. \tag{2.13}$$

The $\bar{u}_i^\alpha$ can be written as functions of x^i, u^α, $u_{(1)}$, meaning that,

$$\bar{u}_i^\alpha = \Phi^\alpha(x^i, u^\alpha, u_{(1)}, \epsilon), \quad \Phi^\alpha\Big|_{\epsilon=0} = u_i^\alpha \tag{2.14}$$

Definition 2.6 The transformations in (2.5) and (2.14) give the first prolongation group $G^{[1]}$.

Definition 2.7 Infinitesimal transformation of the first derivatives is

$$\bar{u}_i^\alpha \approx u_i^\alpha + \zeta_i^\alpha \epsilon, \quad \text{where} \quad \zeta_i^\alpha = \zeta_i^\alpha(x^i, u^\alpha, u_{(1)}, \epsilon) \tag{2.15}$$

Remark 2.8 In terms of infinitesimal transformations, $\mathcal{G}^{[1]}$ is given by (2.6) and (2.15).

(2) Prolonged generators

Definition 2.9 By the relation (2.12) on $\mathcal{G}^{[1]}$ from 2.6, we obtain [6]

$$D_i(x^j + \xi^j \epsilon)(u_j^\alpha + \zeta_j^\alpha \epsilon) = D_i(u^\alpha + \eta^\alpha \epsilon), \quad \text{which gives} \tag{2.16}$$

$$u_i^\alpha + \zeta_j^\alpha \epsilon + u_j^\alpha \epsilon D_i \xi^j = u_i^\alpha + D_i \eta^\alpha \epsilon, \tag{2.17}$$

and thus

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \tag{2.18}$$

is the first prolongation formula.

Remark 2.10 Analogously, one constructs higher order prolongations [6],

$$\zeta_{ij}^\alpha = D_j(\zeta_i^\alpha) - u_{i\kappa}^\alpha D_j(\xi^\kappa), \quad \ldots, \quad \zeta_{i_1,\ldots,i_\kappa}^\alpha = D_{i_\kappa}(\zeta_{i_1,\ldots,i_{\kappa-1}}^\alpha) - u_{i_1,i_2,\ldots,i_{\kappa-1}j}^\alpha D_{i_\kappa}(\xi^j). \tag{2.19}$$

Remark 2.11 The prolonged generators of the prolongations $\mathcal{G}^{[1]}, \ldots, \mathcal{G}^{[\kappa]}$ of the group G are

$$X^{[1]} = X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha}, \quad \ldots, X^{[\kappa]} = X^{[\kappa-1]} + \zeta_{i_1,\ldots,i_\kappa}^\alpha \frac{\partial}{\partial \zeta_{i_1,\ldots,i_\kappa}^\alpha}, \quad \kappa \geq 1, \tag{2.20}$$

for the group generator X in (2.9).

Group invariants.

Definition 2.12 A function $\Gamma(x^i, u^\alpha)$ is said to be an invariant of $\mathcal{G}$ of in (2.1) if

$$\Gamma(\bar{x}^i, \bar{u}^\alpha) = \Gamma(x^i, u^\alpha). \tag{2.21}$$

Theorem 2.13 *A function $\Gamma(x^i, u^\alpha)$ is an invariant of the group $\mathcal{G}$ given by (2.1) if and only if it solves the following first-order linear PDE:* [6]

$$X\Gamma = \xi^i(x^i, u^\alpha)\frac{\partial \Gamma}{\partial x^i} + \eta^\alpha(x^i, u^\alpha)\frac{\partial \Gamma}{\partial u^\alpha} = 0. \tag{2.22}$$

From Theorem (2.13), we have the following result.

Theorem 2.14 *The Lie group $\mathcal{G}$ in (2.1)* [6] *has precisely* n−1 *functionally independent invariants and one can take as the basic invariants, the left-hand sides of the first integrals*

$$\psi_1(x^i, u^\alpha) = c_1, \ldots, \psi_{n-1}(x^i, u^\alpha) = c_{n-1}, \tag{2.23}$$

of the characteristic equations for (2.22):

$$\frac{\mathrm{d}x^i}{\xi^i(x^i, u^\alpha)} = \frac{\mathrm{d}u^\alpha}{\eta^\alpha(x^i, u^\alpha)}. \tag{2.24}$$

Symmetry groups.

Definition 2.15 We define the vector field X (2.9) as a Lie point symmetry of (2.2) if the determining equations

$$X^{[\pi]}\Delta_\alpha\Big|_{\Delta_\alpha=0} = 0, \quad \alpha = 1, \ldots, m, \quad \pi \geq 1, \tag{2.25}$$

are satisfied for the π-th prolongation of X, namely $X^{[\pi]}$.

Definition 2.16 The Lie group $\mathcal{G}$ is a symmetry group of (2.2) if (2.2) is form-invariant, that is

$$\Delta_\alpha\left(\bar{x}^i, \bar{u}^\alpha, \bar{u}_{(1)}, \ldots, \bar{u}_{(\pi)}\right) = 0. \tag{2.26}$$

Theorem 2.17 *The Lie group $\mathcal{G}$ (2.1) can be constructed from the infinitesimal transformations in (2.5) by integrating the Lie equations*

$$\frac{\mathrm{d}\bar{x}^i}{\mathrm{d}\epsilon} = \xi^i(\bar{x}^i, \bar{u}^\alpha), \quad \bar{x}^i\Big|_{\epsilon=0} = x^i, \quad \frac{\mathrm{d}\bar{u}^\alpha}{\mathrm{d}\epsilon} = \eta^\alpha(\bar{x}^i, \bar{u}^\alpha), \quad \bar{u}^\alpha\Big|_{\epsilon=0} = u^\alpha. \tag{2.27}$$

Lie algebras.

Definition 2.18 A vector space V_r of operators [6] X (2.9) is a Lie algebra if for any $X_i, X_j \in V_r$,

$$[X_i, X_j] = X_iX_j - X_jX_i, \tag{2.28}$$

is in V_r for all $i, j = 1, \ldots, r$.

Remark 2.19 The commutator is bilinear, skew symmetric and admits to the Jacobi identity [6].

Theorem 2.20 *The set of solutions of (2.25) forms a Lie algebra* [6].

Exact solutions. The methods of (G'/G)-expansion method [21], Extended Jacobi elliptic function expansion [22] and Kudryashov [19] are usually applied after symmetry reductions.

Conservation laws. [6]
Fundamental operators.

Definition 2.21 The Euler-Lagrange operator $\delta/\delta u^\alpha$ is

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{\kappa \geq 1}(-1)^\kappa D_{i_1}, \ldots, D_{i_\kappa}\frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_\kappa}}, \tag{2.29}$$

and the Lie- Bäcklund operator in abbreviated form [6] is

$$X = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha} + \ldots. \tag{2.30}$$

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Remark 2.22 The Lie- Bäcklund operator (2.30) in its prolonged form is

$$X = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha} + \sum_{\kappa \geq 1}\zeta_{i_1 \ldots i_\kappa}\frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_\kappa}}, \tag{2.31}$$

For

$$\zeta_i^\alpha = D_i(W^\alpha) + \xi^j u_{ij}^\alpha, \qquad \ldots, \zeta_{i_1\ldots i_\kappa}^\alpha = D_{i_1\ldots i_\kappa}(W^\alpha) + \xi^j u_{ji_1\ldots i_\kappa}^\alpha, \quad j = 1, \ldots, n. \tag{2.32}$$

and the Lie characteristic function

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha. \tag{2.33}$$

Remark 2.23 The characteristic form of Lie- Bäcklund operator (2.31) is

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + D_{i_1\ldots i_\kappa}(W^\alpha) \frac{\partial}{\partial u_{i_1 i_2\ldots i_\kappa}^\alpha}. \tag{2.34}$$

The method of multipliers

Definition 2.24 A function $\Lambda^\alpha\left(x^i, u^\alpha, u_{(1)}, \ldots\right) = \Lambda^\alpha$,is a multiplier of (2.2) if [21]

$$\Lambda^\alpha \Delta_\alpha = D_i T^i, \tag{2.35}$$

where $D_i T^i$ is a divergence expression.

Definition 2.25 To find the multipliers Λ^α, one solves the determining equations (2.36) [20],

$$\frac{\delta}{\delta u^\alpha}(\Lambda^\alpha \Delta_\alpha) = 0. \tag{2.36}$$

Ibragimov's conservation theorem . The technique [6] enables one to construct conserved vectors associated with each Lie point symmetry of (2.2).

Definition 2.26 The adjoint equations of (2.2) are

$$\Delta_\alpha^*\left(x^i, u^\alpha, v^\alpha, \ldots, u_{(\pi)}, v_{(\pi)}\right) \equiv \frac{\delta}{\delta u^\alpha}(v^\beta \Delta_\beta) = 0, \tag{2.37}$$

for a new dependent variable v^α.

Definition 2.27 The Formal Lagrangian L of (2.2) and its adjoint equations (2.37) is [6]

$$\mathcal{L} = v^\alpha \Delta_\alpha(x^i, u^\alpha, u_{(1)}, \ldots, u_{(\pi)}). \tag{2.38}$$

Theorem 2.28 *Every infinitesimal symmetry X of (2.2) leads to conservation laws* [6]

$$D_i T^i \Big|_{\Delta_\alpha = 0} = 0, \tag{2.39}$$

where the conserved vector

$$T^i = \xi^i \mathcal{L} + W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j\left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha}\right) + D_j D_k\left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\right) - \ldots\right] + D_j(W^\alpha)\left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k\left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\right) + \ldots\right] + D_j D_k(W^\alpha)\left[\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - \ldots\right]. \tag{2.40}$$

3. Main results

3.1. Lie point symmetries of one-dimensional heat equation (1.1). We start first by computing Lie point symmetries of the one-dimensional heat Equation (1.1), which admits the one-parameter Lie group of transformations with infinitesimal generator

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{3.1}$$

if and only if

$$X^{[2]}\Delta\Big|_{\Delta=0} = 0. \tag{3.2}$$

By using the second prolongation of X, that is, $X^{[2]}$, we obtain

$$\left(\tau\frac{\partial}{\partial t}+\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial u}+\zeta_1\frac{\partial}{\partial u_t}+\zeta_{22}\frac{\partial}{\partial u_{xx}}\right)(u_t-hu_{xx})\Big|_{u_t-hu_{xx}=0}=0 \tag{3.3}$$

which gives

$$\zeta_1-h\zeta_{22}\big|_{u_t=hu_{xx}}=0, \tag{3.4}$$

Where

$$\begin{aligned}
\zeta_1 &= \eta_t+u_t(\eta_u-\tau_t)+u_x(-\xi_t)+u_tu_x(-\xi_u)+u_t^2(-\tau_u),\\
\zeta_2 &= \eta_x+u_x(\eta_u-\xi_x)+u_t(-\tau_x)+u_tu_x(-\tau_u)+u_x^2(-\xi_u),\\
\zeta_{22} &= \eta_{xx}+u_x(2\eta_{xu}-\xi_{xx})+u_t(-\tau_{xx})+u_tu_x(-2\tau_{ux})+u_tu_{xx}(-\tau_u)\\
&\quad +u_{tx}(-2\tau_x)+u_{xx}(\eta_u-2\xi_x)+u_xu_{tx}(-2\tau_u)+u_xu_{xx}(-3\xi_u)+u_x^2(\eta_{uu}-2\xi_{xu})\\
&\quad +u_tu_x^2(-\tau_{uu})+u_x^3(-\xi_{uu}).
\end{aligned} \tag{3.5}$$

If we substitute for ζ_1 and ζ_{22} in the determining Equation (3.4), we obtain the following;

$$\begin{aligned}
&\{\eta_t+u_t\eta_u-u_t\tau_t-u_t^2\tau_u-u_x\xi_t-u_tu_x\xi_u\}\\
&-h\{\eta_{xx}+2u_x\eta_{xu}+u_{xx}\eta_u+u_x^2\eta_{uu}-2u_{xx}\xi_x-u_x\xi_{xx}-2u_x^2\xi_{xu}-3u_xu_{xx}\xi_u\\
&-u_x^3\xi_{uu}-2u_{tx}\tau_x-u_t\tau_{xx}-2u_tu_x\tau_{xu}-(u_tu_{xx}+2u_xu_{tx})\tau_u-u_tu_x^2\tau_{uu}\}\big|_{u_t=hu_{xx}}=0
\end{aligned} \tag{3.6}$$

Now replacing u_{xx} by ut/h in the above equation we obtain,

$$\begin{aligned}
&\{\eta_t+u_t\eta_u-u_t\tau_t-u_t^2\tau_u-u_x\xi_t-u_tu_x\xi_u\}\\
&-h\{\eta_{xx}+2u_x\eta_{xu}+\left[\frac{u_t}{h}\right]\eta_u+u_x^2\eta_{uu}-2\left[\frac{u_t}{h}\right]\xi_x-u_x\xi_{xx}-2u_x^2\xi_{xu}-3u_x\left[\frac{u_t}{h}\right]\xi_u\\
&-u_x^3\xi_{uu}-2u_{tx}\tau_x-u_t\tau_{xx}-2u_tu_x\tau_{xu}-(u_t\left[\frac{u_t}{h}\right]+2u_xu_{tx})\tau_u-u_tu_x^2\tau_{uu}\}=0
\end{aligned} \tag{3.7}$$

Or

$$\begin{aligned}
&\{\eta_t-h\eta_{xx}\}+u_t\{2\xi_x-\tau_t-h\tau_{xx}\}+u_x\{h\xi_{xx}-\xi_t-2h\eta_{xu}\}+u_tu_x\{2\xi_u-2h\tau_{xu}\}\\
&u_x^2\{2h\xi_{xu}-h\eta_{uu}\}+u_x^3\{-h\xi_{uu}\}+u_{tx}\{-2h\tau_x\}+u_x^2\{h\tau_u\}+u_tu_{tx}\{2h\tau_u\}+u_tu_x^2\{h\tau_{uu}\}=0
\end{aligned} \tag{3.8}$$

Now that the functions τ, ξ and η are only of t, x and u and are independent of the derivatives of u, we can then split Equation (3.8) on the derivatives of u and obtain

$$\tau_x=\tau_u=\xi_u=\eta_{uu}=0, \tag{3.9}$$

$$2\xi_x-\tau_t=0, \tag{3.10}$$

$$h\xi_{xx}-2h\eta_{xu}-\xi_t=0 \tag{3.11}$$

$$\eta_t-h\eta_{xx}=0. \tag{3.12}$$

From Equation (3.9), it is evident that

$$\tau=\tau(t), \tag{3.13}$$

$$\xi=\xi(t,x), \tag{3.14}$$

$$\eta=A(t,x)u+B(t,x). \tag{3.15}$$

By making ξ_x, the subject in Equation (3.10), and integrating with respect to x, we have

$$\xi(t,x)=\frac{\tau_t}{2}x+a(t) \tag{3.16}$$

Consequently,

$$\xi_{xx}=0,$$

and Equation (3.11),

$$\xi_t + 2h\eta_{xu} = 0. \tag{3.17}$$

Equation (3.16) is necessary and sufficient for

$$\xi_t = \frac{\tau_t t}{2}x + a_t(t). \tag{3.18}$$

Equation (3.15), also implies that

$$\eta_{xu} = A_x(t,x). \tag{3.19}$$

Now, by Equation (3.18) and (3.19), we have

$$A_x(t,x) = -\frac{\tau_{tt}}{4h}x - \frac{a_t(t)}{2h}, \tag{3.20}$$

which is integrated with respect to x to give

$$A(t,x) = -\frac{\tau_{tt}}{8h}x^2 - \frac{a_t(t)}{2h}x + b(t). \tag{3.21}$$

If we use the values

$$\begin{aligned}\xi =& \frac{\tau_t}{2}x + a(t),\\ \tau =& \tau(t),\\ \eta =& \{-\frac{\tau_{tt}}{8h}x^2 - \frac{a_t(t)}{2h}x + b(t)\}u + B(t,x),\end{aligned} \tag{3.22}$$

in Equation (3.12), we have

$$\{-\frac{\tau_{ttt}}{8h}x^2 - \frac{a_{tt}(t)}{2h}x + b_t(t)\}u + B_t(t,x) + \frac{\tau_{tt}}{4}u - hB_{xx}(t,x) = 0 \tag{3.23}$$

If we separate Equation (3.23), on powers of u yields;

$$u: \ -\frac{\tau_{ttt}}{8h}x^2 - \frac{a_{tt}(t)}{2h}x + b_t(t) + \frac{\tau_{tt}}{4} = 0 \tag{3.24}$$

$$u^0: B_t(t,x) - hB_{xx}(t,x) = 0. \tag{3.25}$$

The solution to Equation (3.25) is any arbitrary function $B(t, x)$ that satisfies one dimensional heat equation (1.1). We can separate Equation (3.24) in powers of x to obtain

$$x^2: \ \frac{\tau_{ttt}(t)}{8h} = 0, \tag{3.26}$$

$$x: \ \frac{a_{tt}(t)}{2h} = 0, \tag{3.27}$$

$$x^0: b_t(t) + \frac{\tau_{tt}}{4} = 0. \tag{3.28}$$

Equations (3.26, 3.27) and (3.28) are solved by

$$\tau = 4hc_1t^2 + 8hc_2t + c_3, \tag{3.29}$$

$$a(t) = 2hc_4t + c_5, \tag{3.30}$$

$$b(t) = -2hc_1t + c_6. \tag{3.31}$$

and finally;

$$\tau = 4hc_1t^2 + 8hc_2t + c_3, \tag{3.32}$$

$$\xi = 4hc_1tx + 4hc_2x + 2hc_4t + c_5, \tag{3.33}$$

$$\eta = -c_1u(2ht + x^2) - c_4xu + c_6u + B(t, x). \tag{3.34}$$

We have obtained an infinite-dimensional Lie algebra of symmetries spanned by

$$X_1 = 4ht^2\frac{\partial}{\partial t} + 4htx\frac{\partial}{\partial x} - u(2ht + x^2)\frac{\partial}{\partial u}, \tag{3.35}$$

$$X_2 = 8ht\frac{\partial}{\partial t} + 4hx\frac{\partial}{\partial x}, \tag{3.36}$$

$$X_3 = \frac{\partial}{\partial t}, \tag{3.37}$$

$$X_4 = 2ht\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}, \tag{3.38}$$

$$X_5 = \frac{\partial}{\partial x}, \tag{3.39}$$

$$X_6 = u\frac{\partial}{\partial u}, \tag{3.40}$$

$$X_\infty = B(t, x)\frac{\partial}{\partial u}. \tag{3.41}$$

Remark 3.1 The one-dimensional heat Equation (1.1) has an infinite-dimensional Lie algebra of point symmetries and many higher symmetries. This is evident from the presence of an arbitrary function of the independent variables in the last symmetry.

3.2 Commutator Table for Symmetries. We evaluate the commutation relations for the symmetry generators. By definition of Lie bracket [22], for example, we have that

$$[X_5, X_3] = X_5X_3 - X_3X_5 = \left(\frac{\partial}{\partial x}\frac{\partial}{\partial t}\right) - \left(\frac{\partial}{\partial t}\frac{\partial}{\partial x}\right) = 0. \tag{3.42}$$

Remark 3.2 The remaining commutation relations are obtained analogously. We present all commutation relations in table (1) below.

Table 1: A commutator table for Lie algebra of one-dimensional heat equation

$[X_i,X_j]$	X_1	X_2	X_3	X_4	X_5	X_6	X_∞
X_1	0	$-8hX_1$	$-X_2 + 2hX_6$	0	$-2X_4$	0	X_{∞_1}
X_2	$8hX_1$	0	$-8hX_3$	$4hX_4$	$-4hX_5$	0	X_{∞_2}
X_3	$X_2 - 2hX_6$	$8hX_3$	0	$2hX_5$	0	0	X_{∞_t}
X_4	0	$-4hX_4$	$-2hX_5$	0	X_6	0	X_{∞_3}
X_5	$2X_4$	$4hX_5$	0	$-X_6$	0	0	X_{∞_x}
X_6	0	0	0	0	0	0	$-X_\infty$
X_∞	$-X_{\infty_1}$	$-X_{\infty_2}$	$-X_{\infty_t}$	$-X_{\infty_3}$	$-X_{\infty_x}$	X_∞	0

Where

$$X_{\infty_1} = 4ht^2X_{\infty_t} + 4htxX_{\infty_x} + (2ht + x^2)X_\infty,$$
$$X_{\infty_2} = 8htX_{\infty_t} + 4hxX_{\infty_x},$$
$$X_{\infty_3} = 2htX_{\infty_x} + xX_\infty.$$

3.3 Group Transformations The corresponding one-parameter group of transformations can be determined by solving the Lie equations [23]. Let $T\epsilon_i$ be the group of transformations for each X_i, i = 1, 2, 3, 4, 5, 6, ∞. We display how to obtain $T\epsilon_i$ from X_i by finding one-parameter group for the infinitesimal generator X_5, namely,

$$X_5 = \frac{\partial}{\partial x}. \tag{3.43}$$

In particular, we have the Lie equations

$$\begin{aligned}\frac{\mathrm{d}\bar{t}}{\mathrm{d}\epsilon_5} =&0, \quad \bar{t}\Big|_{\epsilon_5=0} = t,\\ \frac{\mathrm{d}\bar{x}}{\mathrm{d}\epsilon_5} =&1, \quad \bar{x}\Big|_{\epsilon_5=0} = x,\\ \frac{\mathrm{d}\bar{u}}{\mathrm{d}\epsilon_5} =&0, \quad \bar{u}\Big|_{\epsilon_5=0} = u.\end{aligned} \tag{3.44}$$

Solving the system (3.44) one obtains,

$$\bar{t} = t, \quad \bar{x} = x + \epsilon_5, \quad \bar{u} = u, \tag{3.45}$$

and hence the one-parameter group $T\epsilon_5$ corresponding to the operator X_5 is

$$T_{\epsilon_5}: \quad (\bar{t}, \bar{x}, \bar{u}) = (t, x + \epsilon_5, u). \tag{3.46}$$

All the five one-parameter groups are presented below:

$$\begin{aligned}T_{\epsilon_1}: &\quad (\bar{t}, \bar{x}, \bar{u}) = \left(\frac{t}{1 - 4h\epsilon_1 t}, xe^{4h\epsilon_1 t}, ue^{-(x^2+2ht)\epsilon_1}\right)\\ T_{\epsilon_2}: &\quad (\bar{t}, \bar{x}, \bar{u}) = (te^{8h\epsilon_2}, xe^{4h\epsilon_2}, u)\\ T_{\epsilon_3}: &\quad (\bar{t}, \bar{x}, \bar{u}) = (t + \epsilon_3, x, u)\\ T_{\epsilon_4}: &\quad (\bar{t}, \bar{x}, \bar{u}) = (t, x + 2h\epsilon_4 t, ue^{-\epsilon_4 x}).\\ T_{\epsilon_5}: &\quad (\bar{t}, \bar{x}, \bar{u}) = (t, x + \epsilon_5, u).\\ T_{\epsilon_6}: &\quad (\bar{t}, \bar{x}, \bar{u}) = (t, x, ue^{\epsilon_6}).\\ T_{\epsilon_\infty}: &\quad (\bar{t}, \bar{x}, \bar{u}) = \big(t, x, u + B(t, x)\epsilon_\infty\big).\end{aligned} \tag{3.47}$$

3.4 Symmetry transformations. We now show how the symmetries we have obtained can be used to transform special exact solutions of the one-dimensional heat equation into new solutions. The Lie group analysis vouches for fundamental ways of constructing exact solutions of PDEs, that is, group transformations of known solutions and construction of group-invariant solutions. We will illustrate these methods with examples. If $\bar{u} = g(\bar{t}, \bar{x})$ is a solution of equation (1.1)

$$\phi(t, x, u, \epsilon) = g(f_1(t, x, u, \epsilon), f_2(t, x, u, \epsilon)) \tag{3.48}$$

is also a solution. The one parameter groups dictate to the following generated solutions:

$$\begin{aligned}T_{\epsilon_1}: u =& g\left(\frac{t}{1 - 4h\epsilon_1 t}, xe^{4h\epsilon_1 t}\right)e^{(x^2+2ht)\epsilon_1}\\ T_{\epsilon_2}: u =& g(te^{8h\epsilon_2}, xe^{4h\epsilon_2})\\ T_{\epsilon_3}: u =& g(t + \epsilon_3, x)\\ T_{\epsilon_4}: u =& g(t, x + 2h\epsilon_4 t)e^{\epsilon_4 x}.\\ T_{\epsilon_5}: u =& g(t, x + \epsilon_5).\\ T_{\epsilon_6}: u =& g(t, x)e^{-\epsilon_6}.\\ T_{\Gamma}: u =& g(t, x) - B(t, x)\epsilon_\infty.\end{aligned} \tag{3.49}$$

3.5 Construction of Group-Invariant Solutions. Now we compute the group invariant solutions of one dimensional heat equation.

(i) $X_1 = 4ht^2\frac{\partial}{\partial t} + 4htx\frac{\partial}{\partial x} - u(x^2 + 2ht)\frac{\partial}{\partial u}$

The associated Lagrangian equations

$$\frac{\mathrm{d}t}{4ht^2} = \frac{\mathrm{d}x}{4htx} = \frac{\mathrm{d}u}{-u(x^2 + 2ht)}, \tag{3.50}$$

yield two invariants, $J_1 = x/t$ from $\mathrm{d}t/4ht^2 = \mathrm{d}x/4htx$ and $J_2 = \ln|u| + 1\ 2 \ln t - x^2/4ht$ from $\mathrm{d}t/4ht^2 = \mathrm{d}u/(-u(x^2+2ht))$. Thus using $J_2 = \Phi(J_1)$, we have

$$\ln|u| + \frac{1}{2}\ln t - \frac{x^2}{4ht} = \Phi\left(\frac{x}{t}\right) + C_1, \quad t > 0, \tag{3.51}$$

or

$$u(t, x) = C_2 e^{\frac{4ht\Phi\left(\frac{x}{t}\right) + x^2 - 2ht\ln t}{4ht}}, \quad C_2 = e^{C_1}. \tag{3.52}$$

The derivatives are given by:

$$\begin{aligned} u_t =& C_2 e^{\frac{4ht\Phi\left(\frac{x}{t}\right)+x^2-2ht\ln t}{4ht}}\left\{-\frac{4hx\Phi'\left(\frac{x}{t}\right)+x^2+2ht}{4ht^2}\right\}, \\ u_x =& C_2 e^{\frac{4ht\Phi\left(\frac{x}{t}\right)+x^2-2ht\ln t}{4ht}}\left\{\frac{x+2h\Phi'\left(\frac{x}{t}\right)}{2ht}\right\}, \\ u_{xx} =& C_2 e^{\frac{4ht\Phi\left(\frac{x}{t}\right)+x^2-2ht\ln t}{4ht}}\left\{\frac{(x+2h\Phi')^2+2ht+4h^2\Phi''}{4h^2t^2}\right\}. \end{aligned}$$

If we substitute these derivatives into Equation (1.1), we obtain the first order ordinary differential equation

$$(x^2+2ht)+4hx\Phi'+2h^2\Phi'^2+2h^2\Phi''=0, \tag{3.53}$$

which can be solved and the required group-invariant solution to Equation (1.1) is given by

$$u(t,x)=C_2 e^{\frac{4ht\Phi\left(\frac{x}{t}\right)+x^2-2ht\ln t}{4ht}}. \tag{3.54}$$

(ii). $X_2=8ht\frac{\partial}{\partial t}+4hx\frac{\partial}{\partial x}$

$$\frac{\mathrm{d}t}{8ht}=\frac{\mathrm{d}x}{4hx}=\frac{\mathrm{d}u}{0}. \tag{3.55}$$

This gives the constants $J_1 = u$ and $J_2 = x^2/t$, giving the solution

$$u=\varphi\left(\frac{x^2}{t}\right). \tag{3.56}$$

We obtain the derivatives as follows:

$$u_t=-\frac{x^2}{t^2}\varphi'\left(\frac{x^2}{t}\right), \tag{3.57}$$

$$u_x=\frac{2x}{t}\varphi'\left(\frac{x^2}{t}\right) \tag{3.58}$$

$$u_{xx}=\frac{2}{t}\varphi'\left(\frac{x^2}{t}\right)+\frac{4x^2}{t^2}\varphi''\left(\frac{x^2}{t}\right) \tag{3.59}$$

If we substitute the above derivatives in Equation (1.1), we obtain the second order ordinary differential equation

$$(x^2+2ht)\varphi'+4x^2h\varphi''=0. \tag{3.61}$$

Integrate once with respect to ϕ to obtain

$$(x^2+2ht)\varphi+4x^2h\varphi'=0, \tag{3.62}$$

where we have set the constant of integration to zero. We can write Equation (3.62) as

$$\frac{\mathrm{d}\varphi}{\varphi}=-\frac{\xi+2h}{4h\xi}\mathrm{d}\xi, \quad \xi=\frac{x^2}{t}, \tag{3.63}$$

and on integration, we obtain

$$\ln|\varphi|=-\frac{\xi+2h\ln\xi}{4h}\xi+C_1. \tag{3.64}$$

Then

$$\varphi=C_2 e^{-\frac{x^2+2ht}{4ht}}, \quad C_2=e^{C_1}, \tag{3.65}$$

the solution is

$$u(t,x) = C_2 e^{-\left\{\frac{x^2+4ht\ln x-2ht\ln t}{4ht}\right\}}. \tag{3.66}$$

(iii) $X_3 = \frac{\partial}{\partial t}$ (Stationary solutions)

The Lagrangian system associated with the operator X_3 is

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{0} = \frac{\mathrm{d}u}{0}, \tag{3.67}$$

whose invariants are $J_1 = x$ and $J_2 = u$. So, $u = \psi(x)$ is the group-invariant solution. Substituting of $u = \psi(x)$ into (1.1) yields

$$\psi''(x) = 0. \tag{3.68}$$

Equation (3.68) is a second order linear ODE which is satisfied by the function

$$\psi(x) = C_1 x + C_2. \tag{3.69}$$

Thus, the stationary solution for (1.1) is given by

$$u(t,x) = C_1 x + C_2. \tag{3.70}$$

(iv) $X_4 = 2ht\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}$
Characteristic equations associated to the operator X_4 are

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{2ht} = \frac{\mathrm{d}u}{-xu}, \tag{3.71}$$

Yields $J_1 = t$ and $J_2 = x^2/4ht + \ln|u|$. As a result, the group-invariant solution of (1.1) for this case is $J= = \varphi(J_1)$, for φ an arbitrary function. That is

$$u(t,x) = e^{\frac{4ht\phi(t)-x^2}{4ht}}. \tag{3.72}$$

Now

$$\begin{aligned} u_t &- e^{\frac{4ht\phi(t)-x^2}{4ht}}\left\{\frac{4ht^2\phi'(t)+x^2}{4ht^2}\right\} \\ u_x &= e^{\frac{4ht\phi(t)-x^2}{4ht}}\left\{\frac{-x}{2ht}\right\} \\ u_{xx} &= e^{\frac{4ht\phi(t)-x^2}{4ht}}\left\{\frac{x^2-2ht}{4h^2t^2}\right\} \end{aligned} \tag{3.73}$$

Substitution of the value of u from equation (3.72) into equation (1.1) yields a first order ordinary differential equation

$$2t\phi'(t) + 1 = 0 \tag{3.74}$$

whose general solution is $\varphi(t) = (-\ln|t|+C_6)/2$. Hence, the group-invariant solution under X_4 is

$$u(t,x) = C_7 e^{\frac{-\{x^2+2ht(\ln t+C_6)\}}{4ht}}. \tag{3.75}$$

(v) Space translation -invariant solutions
We consider the space translation operator

$$X_5 = \frac{\partial}{\partial x}. \tag{3.76}$$

Characteristic equations associated with the operator (3.76) are

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{1} = \frac{\mathrm{d}u}{0}, \tag{3.77}$$

which give two invariants $J_1 = t$ and $J_2 = u$. Therefore, $u = \psi(t)$ is the group-invariant solution for some arbitrary function ψ. Substitution of $u = \psi(t)$ into (1.1) yields

$$\psi'(t) = 0, \tag{3.78}$$

whose solution is

$$\psi(t) = C_1, \tag{3.79}$$

for C1 an arbitrary constant. Hence the group-invariant solution of (1.1) under the space translation operator (3.76) is

$$u(t, x) = C_1. \tag{3.80}$$

(vi) $X_6 = u\frac{\partial}{\partial u}$

This Lie point symmetry does not have any invariant solution.

(vii) X_∞

This Lie point symmetry does not have any invariant solution.

3.6 Soliton. We obtain a traveling wave solution for the one-dimensional heat Equation (1.1) by considering a linear combination of the symmetries X_5 and X_3, namely, [21]

$$X = cX_5 + X_3 = c\frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \quad \text{for some constant } c. \tag{3.81}$$

The characteristic equations are

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{c} = \frac{\mathrm{d}u}{0} \tag{3.82}$$

We get two invariants, $J_1 = x - ct$ and $J_2 = u$. So, the group-invariant solution is

$$u(t, x) = \Phi(x - ct), \tag{3.83}$$

for some arbitrary function Φ and c the velocity of the wave. Substitution of u into (1.1) yields a second order ordinary differential equation

$$c\Phi' + h\Phi'' = 0, \tag{3.84}$$

with constant coefficients. If $z = x - ct$ and $\Phi'(z) = y$, then we have a simplified ordinary differential equation of the form

$$cy + hy' = 0, \tag{3.85}$$

whose solution is

$$\Phi'(z) = y = C_7 e^{-\frac{cz}{h}}. \tag{3.86}$$

Thus

$$\Phi(z) = \frac{h}{c}C_7 e^{\frac{-cz}{h}} + C_8. \tag{3.87}$$

Clearly,

$$u(t, x) = C_9 e^{\frac{-c(x-ct)}{h}} + C_8, \quad C_9 = -\frac{h}{c}C_7, \tag{3.88}$$

which is a solitary wave.

4. Conservation laws of equation (1.1)

We will employ multipliers in the construction of conservation laws.

4.1 The multipliers. We make use of the Euler-Lagrange operator defined as defined in [23] to look for a zeroth order multiplier $\Lambda = \Lambda(t, x, u)$. The resulting determining equation for computing Λ is

$$\frac{\delta}{\delta u}\left[\Lambda\{u_t - hu_{xx}\}\right] = 0. \tag{4.1}$$

Where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_x^2\frac{\partial}{\partial u_{xx}} + \cdots \tag{4.2}$$

Expansion of Equation (4.1) yields

$$\Lambda_u(u_t - hu_{xx}) - D_t(\Lambda) - hD_x^2(\Lambda) = 0. \quad (4.3)$$

Invoking the total derivatives

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + \cdots, \quad (4.4)$$

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + \ldots. \quad (4.5)$$

on Equation (4.3) produces

$$-2h(\Lambda_{xu})u_x - 2h(\Lambda_u)u_{xx} - h(\Lambda_{uu})u_x^2 - (\Lambda_t + h\Lambda_{xx}) = 0 \quad (4.6)$$

Splitting Equation (4.6) on derivatives of u produces an overdetermined system of four partial differential equations, namely

$$u_x : \Lambda_{xu} = 0, \quad (4.7)$$

$$u_{xx} : \Lambda_u = 0, \quad (4.8)$$

$$u_x^2 : \Lambda_{uu} = 0, \quad (4.9)$$

$$\text{rest} : \Lambda_t + h\Lambda_{xx} = 0 \quad (4.10)$$

Note that Equation (4.8) is sufficient for Equations (4.9) and (4.7) and implies that

$$\Lambda = \Lambda(t, x) \quad (4.11)$$

By substituting Λ(t, x) into Equation (4.10), we obtain the linear heat equation

$$\Lambda_t + h\Lambda_{xx} = 0. \quad (4.12)$$

Equation (4.12) can be solved by separation of variables. If we assume a solution of the form

$$\Lambda(t, x) = X(x)T(t). \quad (4.13)$$

then Equation (4.12) gives

$$X(x)T_t(t) + hX_{xx}(x)T(t) = 0. \quad (4.14)$$

Dividing by $X(x)hT(t) \neq 0$ and introducing the separation constant $-\lambda^2$, we have

$$T_t(t) - \lambda^2 hT = 0 \quad (4.15)$$

$$X_{xx}(x) + \lambda^2 X(x) = 0 \quad (4.16)$$

The solutions to Equations (4.15) and (4.16) are respectively given by

$$T(t) = C_1 e^{\lambda^2 ht} \quad (4.17)$$

$$X(x) = C_2\cos\lambda x + C_3\sin\lambda x \quad (4.18)$$

which implies that

$$\Lambda(t, x) = e^{\lambda^2 ht}[\mathcal{C}_1\cos\lambda x + \mathcal{C}_2\sin\lambda x], \quad \mathcal{C}_1 = C_1C_2, \quad \mathcal{C}_2 = C_1C_3. \quad (4.19)$$

We finally have the solution to Equation (4.12) as

$$\Lambda(t, x) = e^{\lambda^2 ht}[\mathcal{C}_1\cos\lambda x + \mathcal{C}_2\sin\lambda x]. \quad (4.20)$$

Essentially, we extract the two multiplies

$$\Lambda_1 = e^{\lambda^2 ht}\cos\lambda x \quad (4.21)$$

$$\Lambda_2 = e^{\lambda^2 ht}\sin\lambda x. \tag{4.22}$$

Remark 4.1 Recall that a multiplier Λ for Equation (1.1) has the property that for the density $T^t = T^t(t, x, u)$ and flux $T^x = T^x(t, x, u, u_x)$,

$$\Lambda(u_t - hu_{xx}) = D_t T^t + D_x T^x, \tag{4.23}$$

Where

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + \cdots, \tag{4.24}$$

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + \cdots. \tag{4.25}$$

We derive a conservation law corresponding to each of the multipliers.

(i) Conservation law for the multiplier $\Lambda_1 = e^{\lambda^2 ht}\cos\lambda x$

Expansion of equation (4.23) gives

$$e^{\lambda^2 ht}\cos\lambda x\{u_t - hu_{xx}\} = T_t^t + u_t T_u^t + +T_x^x + u_x T_u^x + u_{xx}T_{u_x}^x. \tag{4.26}$$

Splitting Equation (4.26) on the second derivative of u yields

$$u_{xx} \ : \ T_{u_x}^x = -he^{\lambda^2 ht}\cos\lambda x, \tag{4.27}$$

$$\text{Rest} \ : \ e^{\lambda^2 ht}\cos\lambda x\{u_t\} = T_t^t + T_u^t u_t + T_x^x + T_u^x u_x. \tag{4.28}$$

The integration of Equation (4.27) with respect to u_x gives

$$T^x = -hu_x e^{\lambda^2 ht}\cos\lambda x + A(t, x, u). \tag{4.29}$$

Substituting the expression of T^x from (4.29) into Equation (4.26) we get

$$e^{\lambda^2 ht}\cos\lambda x\{u_t\} = T_t^t + T_u^t u_t + A_x(t,x,u) + \{A_u(t,x,u) + \lambda hu_x e^{\lambda^2 t}\sin\lambda x\}u_x \tag{4.30}$$

which splits on first derivatives of u, to give

$$u_x \ : \ A_u(t,x,u) = -h\lambda e^{\lambda^2 ht}\sin\lambda x, \tag{4.31}$$

$$u_t \ : \ T_u^t = e^{\lambda^2 ht}\cos\lambda x, \tag{4.32}$$

$$\text{Rest} \ : \ 0 = T_t^t + A_x(t,x,u). \tag{4.33}$$

Integrating equations (4.31) and (4.32) with respect to u manifests that

$$T^t = ue^{\lambda^2 ht}\cos\lambda x + C(t,x), \tag{4.34}$$

$$A(t,x,u) = -h\lambda ue^{\lambda^2 ht}\sin\lambda x + B(t,x) \tag{4.35}$$

By substituting the obtained functions into Equation (4.30), we have

$$C_t(t,x) + B_x(t,x) = 0. \tag{4.36}$$

Since $C(t, x)$ and $B(t, x)$ contribute to the trivial part of the conservation law, we take $C(t, x) = B(t, x) = 0$ and obtain the conserved quantities

$$T^t = ue^{\lambda^2 ht}\cos\lambda x, \tag{4.37}$$

$$T^x = -he^{\lambda^2 ht}\{u_x\cos\lambda x + \lambda u\sin\lambda x\} \tag{4.38}$$

from which the conservation law corresponding to the multiplier $\Lambda_1 = e^{\lambda^2 ht}\cos\lambda x$ is given by

$$D_t\{ue^{\lambda^2 ht}\cos\lambda x\} - he^{\lambda^2 ht}D_x\{u_x\cos\lambda x + \lambda u\sin\lambda x\} = 0. \tag{4.39}$$

(ii) Conservation law for the multiplier $\Lambda_2 = e^{\lambda^2 t}\sin\lambda x$

Expansion of equation (4.23) gives

$$e^{\lambda^2 ht}\sin\lambda x\{u_t - hu_{xx}\} = T^t_t + u_t T^t_u + +T^x_x + u_x T^x_u + u_{xx}T^x_{u_x}. \quad (4.40)$$

Splitting Equation (4.40) on the second derivative of u yields

$$u_{xx} \ : \ T^x_{u_x} = -he^{\lambda^2 ht}\sin\lambda x, \quad (4.41)$$

$$\text{Rest} \ : \ e^{\lambda^2 ht}\sin\lambda x = T^t_t + u_t T^t_u + T^x_x + T^x_u u_x. \quad (4.42)$$

The integration of Equation (4.41) with respect to u_x gives

$$T^x = -hu_x e^{\lambda^2 ht}\sin\lambda x + a(t,x,u). \quad (4.43)$$

Substituting the expression of T x from (4.43) into Equation (4.40) we get

$$e^{\lambda^2 ht}\sin\lambda x\{u_t\} = T^t_t + T^t_u u_t - \lambda hu_x e^{\lambda^2 t}\cos\lambda x + a_x(t,x,u) + a_u(t,x,u)u_x. \quad (4.44)$$

which splits on first derivatives of u, to give

$$u_x \ : a_u(t,x,u) = \lambda he^{\lambda^2 ht}\cos\lambda x, \quad (4.45)$$

$$u_t \ : T^t_u = e^{\lambda^2 ht}\sin\lambda x, \quad (4.46)$$

$$\text{Rest} \ : 0 = T^t_t + a_x(t,x,u). \quad (4.47)$$

Integrating equations (4.45) and (4.46) with respect to u manifests that

$$T^t = ue^{\lambda^2 ht}\sin\lambda x + c(t,x), \quad (4.48)$$

$$a(t,x,u) = \lambda hue^{\lambda^2 ht}\cos\lambda x + b(t,x). \quad (4.49)$$

By substituting the obtained functions into Equation (4.44), we have

$$c_t(t,x) + b_x(t,x) = 0. \quad (4.50)$$

We may take $c(t, x)$ and $c(t, x)$ as contributing to the trivial part of the conservation law and set them to $c(t, x) = b(t, x) = 0$ and obtain the conserved quantities

$$T^t = ue^{\lambda^2 ht}\sin\lambda x, \quad (4.51)$$

$$T^x = -he^{\lambda^2 ht}\{u_x\sin\lambda x - \lambda u\cos\lambda x\} \quad (4.52)$$

from which the conservation law corresponding to the multiplier $\Lambda_2 = e^{\lambda^2 ht}\sin\lambda x$ is given by

$$D_t\{ue^{\lambda^2 ht}\sin\lambda x\} - he^{\lambda^2 ht}D_x\{u_x\sin\lambda x - \lambda u\cos\lambda x\} = 0. \quad (4.53)$$

Remark 4.2 It can be shown that the two sets of conserved quantities are conservation laws. Given that $(e^{\lambda^2 ht}, \sin\lambda x, \cos\lambda x) \neq (0,0,0)$, the verification reaffirms that the one-dimensional equation is itself a conversation law.

5. Conclusion

In this manuscript, an infinite dimensional Lie algebra of Lie point symmetries has been applied to study a one-dimensional heat equation. A commutator table has been constructed for the obtained Lie algebra. We have also used symmetry reductions to compute exact group-invariant solutions, including a soliton. Conservation laws have also been derived for the model with the use of zeroth order multipliers.

6. Acknowledgement

The author thanks referees and the editor for their careful reading and comments.

7. Author's contribution

The author wrote the article as a scholarly duty and passion to disseminate mathematical research and hereby declares that there is no conflict of interest.

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