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Group Invariant Solutions and Conserved Vectors for a Special KdV Type Equation

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Abstract
We study a special simple 'kdv type' equation by Lie group analysis. The obtained Lie point symmetries are used to carry out symmetry reductions and the resulting reduced systems investigated for exact solutions. Traveling waves obtained are a linear span of time and space translation symmetries. The multiplier technique has been employed in the construction of conserved vectors for KdV equation.

Keywords: kdv Equations, Symmetries, Group-Invariant Solutions, Stationary Solutions, Symmetry Reductions, Solitons, Traveling Waves

1. Introduction

A Scottish mathematician, John Scott Russell [25] is credited for being the first to work in solitary waves in 1844. Russell observed water waves, set in motion by a boat drawn in Edinburg-Glasgow canal, that maintained shape and structure. He also conducted experiments, which culminated to the discovery of solitons-localized, highly stable waves whose shape and speed are invariant with time. In 1895, Diederik Johannes Korteweg (1848-1941) alongside his PhD student, Gustav de Vries analytically derived the non-linear partial differential equation, present day Korteweg-de Vries equation. However, a prominent French mathematician Joseph Valentin Boussinesq (1842- 1929) had earlier introduced the Korteweg-de Vries equation in 1877 in his work on water waves prior to this development. This elegant equation models the disturbance of the surface of shallow water in the presence of solitary waves. The aforementioned equation in a simple form is given by

$$u_t + auu_x + u_{xxx} = 0 \tag{1.1}$$

and combines non-linearity term uu_x which localizes the wave and linear dispersion term u_{xxx} which spreads it out. Note that $u(x, t)$ denotes the elongation of the wave at place x and time t. The KdV equation shows up in a wide range of physics phenomena, especially those exhibiting shock waves, traveling waves and solitons. In the area of quantum mechanics, some theoretical physics phenomena are explained by means of a Korteweg-de Vries equation model. Some of the numerous applications are in aerodynamics, fluid dynamics, and continuum mechanics as a model for shock wave formation, solitons, turbulence, boundary layer behaviour and mass transport. Indeed, Korteweg-de Vries equation is ubiquitous and very important in the theory of integrable systems given that it is known to possess an infinite number of conservation laws and gives rise to multiple-soliton solutions. Observe that through scaling, we associate any equation of the form

$$\alpha u_t + \beta uu_x + \gamma u_{xxx} = 0, \tag{1.2}$$

to be of "KdV type". The form of Korteweg-de Vries equation used for this manuscript is

$$\Delta \equiv u_t + 6uu_x + u_{xxx} = 0, \tag{1.3}$$

for which the 6 factor is just conventional and of no great significance. This is a special case in which $\alpha = 1$, $\beta = 6$ and $\gamma = 1$

2. Preliminaries

In this section, we outline preliminary concepts which are useful in the sequel.

Local Lie groups

In Euclidean spaces R^n of $x = x^i$ independent variables and R^m of $u = u^\alpha$ dependent variables, we consider the transformations [7]

$$T_\epsilon: \quad \bar{x}^i = \varphi^i(x^i, u^\alpha, \epsilon), \quad \bar{u}^\alpha = \psi^\alpha(x^i, u^\alpha, \epsilon), \tag{2.1}$$

involving the continuous parameter ϵ which ranges from a neighbourhood $\mathcal{N}' \subset \mathcal{N} \subset \mathbb{R}$ of $\epsilon = 0$ where the functions ϕ^i and ψ^α differentiable and analytic in the parameter ϵ.

Definition 2.1: The set $\mathcal{G}$ of transformations given by (2.1) is a local Lie group if it holds true that

1. (Closure) Given $T_{\epsilon1}, T_{\epsilon2} \in \mathcal{G}$, for $\epsilon_1, \epsilon_2 \in$ N J ⊂ N, then
   $T_{\epsilon1}\, T_{\epsilon2} = T_{\epsilon3} \in \mathcal{G}$, $\epsilon_3 = \varphi(\epsilon_1, \epsilon_2) \in$ N .
2. (Identity) There exists a unique $T_0 \in \mathcal{G}$ if and only if $\epsilon = 0$ such that $T_\epsilon T_0 = T_0 T_\epsilon = T_\epsilon$.
3. (Inverse) There exists a unique $T\epsilon{-}1 \in \mathcal{G}$ for every transformation $T_\epsilon \in \mathcal{G}$,
   where $\epsilon \in$ N J ⊂ N and $\epsilon^{-1} \in$ N
   such that $T_\epsilon T\epsilon{-}1 = T\epsilon{-}1\; T_\epsilon = T_0$.

Remark 2.2: Associativity of the group $\mathcal{G}$ in (2.1) follows from (1).

Prolongations

In the system,

$$\Delta_\alpha\left(x^i, u^\alpha, u_{(1)}, \ldots, u_{(\pi)}\right) = \Delta_\alpha = 0, \tag{2.2}$$

the variables u^α are dependent. The partial derivatives $u(1) = \{u^\alpha\}$, $u(2) = \{u^\alpha\ \}$, . . . , $u(\pi) = \{u^\alpha{}_{i1\ldots i\pi}\}$, are of the first, second, . . . , up to the πth-orders.

Denoting

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \ldots, \tag{2.3}$$

the total differentiation operator with respect to the variables x^i *and* $\delta^j{}_i$, the Kronecker delta, we have

$$D_i(x^j) = \delta_i^j, ', \; u_i^\alpha = D_i(u^\alpha), \;\; u_{ij}^\alpha = D_j(D_i(u^\alpha)), \;\; \ldots, \tag{2.4}$$

where u^α defined in (2.4) are differential variables Ibragimov [11].

1. Prolonged groups Consider the local Lie group $\mathcal{G}$ given by the transformations

$$\bar{x}^i = \varphi^i(x^i, u^\alpha, \epsilon), \quad \varphi^i\Big|_{\epsilon=0} = x^i, \;\; \bar{u}^\alpha = \psi^\alpha(x^i, u^\alpha, \epsilon), \quad \psi^\alpha\Big|_{\epsilon=0} = u^\alpha, \tag{2.5}$$

where the symbol $\Big|_{\epsilon=0}$ means evaluated on $\epsilon = 0$.

Definition 2.3: The construction of the group $\mathcal{G}$ given by (2.5) is an equivalence of the computation of infinitesimal transformations

$$\begin{aligned} &\bar{x}^i \approx x^i + \xi^i(x^i, u^\alpha)\epsilon, \quad \varphi^i\Big|_{\epsilon=0} = x^i, \\ &\bar{u}^\alpha \approx u^\alpha + \eta^\alpha(x^i, u^\alpha)\epsilon, \quad \psi^\alpha\Big|_{\epsilon=0} = u^\alpha, \end{aligned} \tag{2.6}$$

obtained from (2.1) by a Taylor series expansion of $\phi^i(x^i, u^\alpha, \epsilon)$ and $\psi^i(x^i, u^\alpha, \epsilon)$ in ϵ about $\epsilon = 0$ and keeping only the terms linear in ϵ, where

$$\xi^i(x^i, u^\alpha) = \frac{\partial \varphi^i(x^i, u^\alpha, \epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}, \quad \eta^\alpha(x^i, u^\alpha) = \frac{\partial \psi^\alpha(x^i, u^\alpha, \epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}. \tag{2.7}$$

Remark 2.4: The symbol of infinitesimal transformations, X, is used to write (2.6) as

$$\bar{x}^i \approx (1+X)x^i, \quad \bar{u}^\alpha \approx (1+X)u^\alpha, \tag{2.8}$$

Where

$$X = \xi^i(x^i, u^\alpha)\frac{\partial}{\partial x^i} + \eta^\alpha(x^i, u^\alpha)\frac{\partial}{\partial u^\alpha}, \tag{2.9}$$

is the generator of the group G given by (2.5).

Remark 2.5: To obtain transformed derivatives from (2.1), we use a change of variable formulae

$$D_i = D_i(\varphi^j)\bar{D}_j, \tag{2.10}$$

where $\bar{D}_j$ is the total differentiation in the variables $\bar{x}^i$. This means that

$$\bar{u}_i^\alpha = \bar{D}_i(\bar{u}^\alpha), \; \bar{u}_{ij}^\alpha = \bar{D}_j(\bar{u}_i^\alpha) = \bar{D}_i(\bar{u}_j^\alpha). \tag{2.11}$$

If we apply the change of variable formula given in (2.10) on G given by (2.5), we get

$$D_i(\psi^\alpha) = D_i(\varphi^j), \; \bar{D}_j(\bar{u}^\alpha) = \bar{u}_j^\alpha D_i(\varphi^j) \tag{2.12}$$

Expansion of (2.12) yields

$$\left(\frac{\partial \varphi^j}{\partial x^i} + u_i^\beta \frac{\partial \varphi^j}{\partial u^\beta}\right)\bar{u}_j^\beta = \frac{\partial \psi^\alpha}{\partial x^i} + u_i^\beta \frac{\partial \psi^\alpha}{\partial u^\beta} \tag{2.13}$$

The variables $\bar{u}^\alpha$ can be written as functions of x^i, u^α, $u(1)$, that is

$$\bar{u}_i^\alpha = \Phi^\alpha(x^i, u^\alpha, u_{(1)}, \epsilon), \quad \Phi^\alpha\Big|_{\epsilon=0} = u_i^\alpha \tag{2.14}$$

Definition 2.6: The transformations in the space of the variables x^i, u^α, $u_{(1)}$ given in (2.5) and (2.14) form the first prolongation group G [1].

Definition 2.7: Infinitesimal transformation of the first derivatives is

$$\bar{u}_i^\alpha \approx u_i^\alpha + \zeta_i^\alpha \epsilon, \quad \text{where} \quad \zeta_i^\alpha = \zeta_i^\alpha(x^i, u^\alpha, u_{(1)}, \epsilon) \tag{2.15}$$

Remark 2.8: In terms of infinitesimal transformations, the first prolongation group $G^{[1]}$ is given by (2.6) and (2.15).

2. Prolonged generators

Definition 2.9: By using the relation given in (2.12) on the first prolongation group $G^{[1]}$ given by Definition 2.6, we obtain [7]

$$D_i(x^j + \xi^j\epsilon)(u_j^\alpha + \zeta_j^\alpha\epsilon) = D_i(u^\alpha + \eta^\alpha\epsilon), \quad \text{which gives} \tag{2.16}$$

$$u_i^\alpha + \zeta_j^\alpha\epsilon + u_j^\alpha\epsilon D_i\xi^j = u_i^\alpha + D_i\eta^\alpha\epsilon, \tag{2.17}$$

and thus

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \tag{2.18}$$

is the first prolongation formula.

Remark 2.10: Similarly, we get higher order prolongations [11],

$$\zeta_{ij}^\alpha = D_j(\zeta_i^\alpha) - u_{i\kappa}^\alpha D_j(\xi^\kappa), \quad \ldots, \quad \zeta_{i_1,\ldots,i_\kappa}^\alpha = D_{i_\kappa}(\zeta_{i_1,\ldots,i_{\kappa-1}}^\alpha) - u_{i_1,i_2,\ldots,i_{\kappa-1}j}^\alpha D_{i_\kappa}(\xi^j). \tag{2.19}$$

Remark 2.11: The prolonged generators of the prolongations $G^{[1]}, \ldots, G^{[\kappa]}$ of the group G are

$$X^{[1]} = X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha}, \ \ldots, X^{[\kappa]} = X^{[\kappa-1]} + \zeta_{i_1,\ldots,i_\kappa}^\alpha \frac{\partial}{\partial \zeta_{i_1,\ldots,i_\kappa}^\alpha}, \ \kappa \geq 1, \quad (2.20)$$

where X is the group generator given by (2.9).

Group invariants

Definition 2.12: A function $\Gamma(x^i, u^\alpha)$ is called an invariant of the group $\mathcal{G}$ of transformations given by (2.1) if

$$\Gamma(\bar{x}^i, \bar{u}^\alpha) = \Gamma(x^i, u^\alpha) \quad (2.21)$$

Theorem 2.13: *A function $\Gamma(x^i, u^\alpha)$ is an invariant of the group $\mathcal{G}$ given by (2.1) if and only if it solves the following first-order linear PDE:* [7]

$$X\Gamma = \xi^i(x^i, u^\alpha)\frac{\partial \Gamma}{\partial x^i} + \eta^\alpha(x^i, u^\alpha)\frac{\partial \Gamma}{\partial u^\alpha} = 0 \quad (2.22)$$

From Theorem (2.13), we have the following result.

Theorem 2.14: *The local Lie group $\mathcal{G}$ of transformations in* Rn *given by (2.1)* [11] *has precisely n - 1 functionally independent invariants. One can take, as the basic invariants, the left-hand sides of the first integrals*

$$\psi_1(x^i, u^\alpha) = c_1, \ldots, \psi_{n-1}(x^i, u^\alpha) = c_{n-1}, \quad (2.23)$$

of the characteristic equations for (2.22):

$$\frac{\mathrm{d}x^i}{\xi^i(x^i, u^\alpha)} = \frac{\mathrm{d}u^\alpha}{\eta^\alpha(x^i, u^\alpha)}. \quad (2.24)$$

Symmetry groups

Definition 2.15: The vector field X (2.9) is a Lie point symmetry of the PDE system (2.2) if the determining equations

$$X^{[\pi]}\Delta_\alpha\Big|_{\Delta_\alpha=0} = 0, \quad \alpha = 1, \ldots, m, \quad \pi \geq 1, \quad (2.25)$$

are satisfied, where $|_{\Delta_\alpha=0}$ means evaluated on $\Delta_\alpha = 0$ and $X^{[\pi]}$ is the π-th prolongation of X.

Definition 2.16. The Lie group $\mathcal{G}$ is a symmetry group of the PDE system given in (2.2) if the PDE system (2.2) is form-invariant, that is

$$\Delta_\alpha\left(\bar{x}^i, \bar{u}^\alpha, \bar{u}_{(1)}, \ldots, \bar{u}_{(\pi)}\right) = 0 \quad (2.26)$$

Theorem 2.17: *Given the infinitesimal transformations in (2.5), the Lie group $\mathcal{G}$ in (2.1) is found by integrating the Lie equations*

$$\frac{\mathrm{d}\bar{x}^i}{\mathrm{d}\epsilon} = \xi^i(\bar{x}^i, \bar{u}^\alpha), \quad \bar{x}^i\Big|_{\epsilon=0} = x^i, \quad \frac{\mathrm{d}\bar{u}^\alpha}{\mathrm{d}\epsilon} = \eta^\alpha(\bar{x}^i, \bar{u}^\alpha), \quad \bar{u}^\alpha\Big|_{\epsilon=0} = u^\alpha \quad (2.27)$$

Lie algebras

Definition 2.18. A vector space V_r of operators [7] X (2.9) is a Lie algebra if for any two opera- tors, X_i, $X_j \in V_r$, their commutator

$$[X_i, X_j] = X_iX_j - X_jX_i, \quad (2.28)$$

is in V_r for all $i, j = 1, \ldots, r$.

Remark 2.19: The commutator satisfies the properties of bilinearity, skew symmetry and the Jacobi identity [7].

Theorem 2.20: *The set of solutions of the determining equation given by (2.25) forms a Lie algebra* [7].

Conservation laws [10]

Fundamental operators

Let a system of πth-order PDEs be given by (2.2).

Definition 2.21: The Euler-Lagrange operator $\delta/\delta u^\alpha$ is

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{\kappa \geq 1} (-1)^\kappa D_{i_1}, \ldots, D_{i_\kappa} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_\kappa}}, \tag{2.29}$$

and the Lie- Ba¨cklund operator in abbreviated form [7] is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \ldots . \tag{2.30}$$

Remark 2.22: The Lie- Ba¨cklund operator (2.30) in its prolonged form is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{\kappa \geq 1} \zeta_{i_1 \ldots i_\kappa} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_\kappa}}, \tag{2.31}$$

Where

$$\zeta^\alpha_i = D_i(W^\alpha) + \xi^j u^\alpha_{ij}, \qquad \ldots, \zeta^\alpha_{i_1 \ldots i_\kappa} = D_{i_1 \ldots i_\kappa}(W^\alpha) + \xi^j u^\alpha_{j i_1 \ldots i_\kappa}, \quad j = 1, \ldots, n \tag{2.32}$$

and the Lie characteristic function is

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j \tag{2.33}$$

Remark 2.23: The characteristic form of Lie- Backlund operator (2.31) is

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + D_{i_1 \ldots i_\kappa}(W^\alpha) \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_\kappa}} \tag{2.34}$$

Remark 2.24: Noether's Theorem is applicable to systems from variational problems

The method of multipliers

Definition 2.25: A function Λ^α x^i, u^α, $u(1)$, . . . = Λ^α, is a multiplier of the PDE system given by (2.2) if it satisfies the condition that [18]

$$\Lambda^\alpha \Delta_\alpha = D_i T^i, \tag{2.35}$$

where $D_i T^i$ is a divergence expression.

Definition 2.26: To find the multipliers Λ^α, one solves the determining equations (2.36) [?],

$$\frac{\delta}{\delta u^\alpha}(\Lambda^\alpha \Delta_\alpha) = 0 \tag{2.36}$$

Ibragimov's conservation theorem

The technique [10] enables one to construct conserved vectors associated with each Lie point symmetry of the PDE system given by (2.2).

Definition 2.27. The adjoint equations of the system given by (2.2) are

$$\Delta^*_\alpha\left(x^i, u^\alpha, v^\alpha, \ldots, u_{(\pi)}, v_{(\pi)}\right) \equiv \frac{\delta}{\delta u^\alpha}(v^\beta \Delta_\beta) = 0, \tag{2.37}$$

where v^α is the new dependent variable.

Definition 2.28: Formal Lagrangian L of the system (2.2) and its adjoint equations (2.37) is [9]

$$\mathcal{L} = v^\alpha \Delta_\alpha(x^i, u^\alpha, u_{(1)}, \ldots, u_{(\pi)}) \tag{2.38}$$

Theorem 2.29: *Every infinitesimal symmetry X of the system given by (2.2) leads to conservation laws* [11]

$$D_i T^i \Big|_{\Delta_\alpha = 0} = 0, \tag{2.39}$$

where the conserved vector

$$T^i = \xi^i \mathcal{L} + W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j \left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} \right) + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) - \dots \right] + D_j(W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) + \dots \right] + D_j D_k (W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - \dots \right]. \tag{2.40}$$

3. Main results

We compute the Lie point symmetries of Equation (1.3) and use them to linearize the problem and construct exact solutions.

3.1 Lie point symmetries of (1.3)

The vector field

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}, \tag{3.1}$$

is a Lie point symmetry of (1.3) if and only if

$$\mathbf{X}^{[3]}\Delta \Big|_{\Delta=0} = 0, \tag{3.2}$$

where t and x are two independent variables and u is a dependent variable. $\mathrm{X}^{[3]}$ is the third prolongation [8] of (1.3) defined by

$$\mathrm{X}^{[3]} = X + \zeta_1 \frac{\partial}{\partial u_t} + \zeta_2 \frac{\partial}{\partial u_x} + \zeta_{222} \frac{\partial}{\partial u_{xxx}}, \tag{3.3}$$

Where

$$\zeta_1 = D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi), \tag{3.4}$$

$$\zeta_2 = D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi), \tag{3.5}$$

$$\zeta_{22} = D_x(\zeta_2) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi), \tag{3.6}$$

$$\zeta_{222} = D_x(\zeta_{22}) - u_{txx} D_x(\tau) - u_{xxx} D_x(\xi), \tag{3.7}$$

and the total derivatives D_t and D_x are given by

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t} + \cdots, \tag{3.8}$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{tx} \frac{\partial}{\partial u_t} + \dots. \tag{3.9}$$

Applying the definitions of D_t and D_x given above, we obtain the expanded form of the ζ_s as

$$\begin{aligned}
\zeta_1 =& \eta_t + u_t \eta_u - u_t \tau_t - u_t^2 \tau_u - u_x \xi_t - u_t u_x \xi_u, \\
\zeta_2 =& \eta_x + u_x \eta_u - u_t \tau_x - u_t u_x \tau_u - u_x \xi_x - u_x^2 \xi_u, \\
\zeta_{22} =& \eta_{xx} + 2u_x \eta_{xu} + u_{xx} \eta_u + u_x^2 \eta_{uu} - 2u_{xx} \xi_x - u_x \xi_{xx} - 2u_x^2 \xi_{xu} - 3u_x u_{xx} \xi_u \\
& - u_x^3 \xi_{uu} - 2u_{tx} \tau_x - u_t \tau_{xx} - 2u_t u_x \tau_{xu} - (u_t u_{xx} + 2u_x u_{tx}) \tau_u - u_t u_x^2 \tau_{uu}, \\
\zeta_{222} =& \eta_{xxx} - 3u_t u_x u_t u_{xx} \tau_{uu} + u_x^3 \eta_{uuu} + 3u_x^2 \eta_{uux} - u_t \tau_{xxx} - 3u_x^2 \xi_{uux} - 3u_x^3 \xi_{uux} \\
& - u_x^4 \xi_{uuu} - 3u_{xx}^2 \xi_u + 3u_{xx} \eta_{ux} + u_{xxx} \eta_u - 3u_{txx} \tau_x - 3u_{tx} \tau_{xx} - 3u_{xxx} \xi_x \\
& - 3u_{xx} \xi_{xx} - 3u_t u_x \tau_{uxx} - 3u_t u_{xx} \tau_{ux} - 6u_{tx} u_x \tau_{ux} + 3u_{xx} u_x \eta_{uu} - 3u_{tx} u_x^2 \tau_{uu} \\
& + 3u_x \eta_{uxx} - u_x \xi_{xxx} - 3u_t u_x^2 \tau_{uux} - u_t u_{xxx} \tau_u - 3u_{txx} u_x \tau_u - u_t u_x^3 \tau_{uuu} \\
& - 4u_{xxx} u_x \xi_u - 9u_x u_{xx} \xi_{ux} - 3u_{tx} u_{xx} \tau_u - 6u_x^2 u_{xx} \xi_{uu}.
\end{aligned} \tag{3.10}$$

From equation (3.2) we get

$$\left[\tau\frac{\partial}{\partial t}+\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial u}+\zeta_1\frac{\partial}{\partial u_t}+\zeta_2\frac{\partial}{\partial u_x}+\zeta_{222}\frac{\partial}{\partial u_{xxx}}\right]\left(u_t+6uu_x+u_{xxx}\right)\Big|_{(1.3))}=0 \tag{3.11}$$

After expanding the above equation we obtain

$$\left(6u_x\eta+\zeta_1+6u\zeta_2+\zeta_{222}\right)\Big|_{u_{xxx}=-u_t-6uu_x}=0 \tag{3.12}$$

Substituting the values of ζ_1, ζ_2 and ζ_{222} in the above equation yields

$$\begin{aligned}&6u_x\eta+[\eta_t+u_t\eta_u-u_t\tau_u-u_t^2\tau_u-u_x\xi_t-u_tu_x\xi_u]+6u[\eta_x+u_x\eta_u-u_t\tau_x-u_tu_x\tau_u\\&-u_x\xi_x-u_x^2\xi_u]+[\eta_{xxx}-3u_tu_xu_tu_{xx}\tau_{uu}+u_x^3\eta_{uuu}+3u_x^2\eta_{uux}-u_t\tau_{xxx}-3u_x^2\xi_{uux}\\&-3u_x^3\xi_{uux}-u_x^4\xi_{uuu}-3u_{xx}^2\xi_u+3u_{xx}\eta_{ux}+u_{xxx}\eta_u-3u_{txx}\tau_x-3u_{tx}\tau_{xx}-3u_{xxx}\xi_x\\&-3u_{xx}\xi_{xx}-3u_tu_x\tau_{uxx}-3u_tu_{xx}\tau_{ux}-6u_{tx}u_x\tau_{ux}+3u_{xx}u_x\eta_{uu}-3u_{tx}u_x^2\tau_{uu}\\&-u_x\xi_{xxx}-3u_tu_x^2\tau_{uux}-u_tu_{xxx}\tau_u-3u_{txx}u_x\tau_u-u_tu_x^3\tau_{uuu}-4u_{xxx}u_x\xi_u-9u_xu_{xx}\xi_{ux}\\&-3u_{tx}u_{xx}\tau_u-6u_x^2u_{xx}\xi_{uu}]\Big|_{u_{xxx}=-u_t-6uu_x}=0.\end{aligned} \tag{3.13}$$

Replacing u_{xxx} by $-u_t-6uu_x$ in the above equation, we obtain

$$\begin{aligned}&6u_x\eta+[\eta_t+u_t\eta_u-u_t\tau_u-u_t^2\tau_u-u_x\xi_t-u_tu_x\xi_u]+6u[\eta_x+u_x\eta_u-u_t\tau_x-u_tu_x\tau_u\\&-u_x\xi_x-u_x^2\xi_u]+[\eta_{xxx}-3u_tu_xu_tu_{xx}\tau_{uu}+u_x^3\eta_{uuu}+3u_x^2\eta_{uux}-u_t\tau_{xxx}-3u_x^2\xi_{uux}\\&-3u_x^3\xi_{uux}-u_x^4\xi_{uuu}-3u_{xx}^2\xi_u+3u_{xx}\eta_{ux}+(-u_t-6uu_x)\eta_u-3u_{txx}\tau_x-3u_{tx}\tau_{xx}\\&-3(-u_t-6uu_x)\xi_x-3u_{xx}\xi_{xx}-3u_tu_x\tau_{uxx}-3u_tu_{xx}\tau_{ux}-6u_{tx}u_x\tau_{ux}+3u_{xx}u_x\eta_{uu}\\&-3u_{tx}u_x^2\tau_{uu}+3u_x\eta_{uxx}-u_x\xi_{xxx}-3u_tu_x^2\tau_{uux}-u_t(-u_t-6uu_x)\tau_u-3u_{txx}u_x\tau_u\\&-u_tu_x^3\tau_{uuu}-4(-u_t-6uu_x)u_x\xi_u-9u_xu_{xx}\xi_{ux}-3u_{tx}u_{xx}\tau_u-6u_x^2u_{xx}\xi_{uu}]=0.\end{aligned} \tag{3.14}$$

Splitting the above determining equation on derivatives of u gives the following over determined system of eight linear partial differential equations, namely,

$$u_{txx}\ :\ \tau_x=0, \tag{3.15}$$

$$u_{tx}u_{xx}\ :\ \tau_u=0, \tag{3.16}$$

$$u_{xx}^2\ :\ \xi_u=0, \tag{3.17}$$

$$u_xu_{xx}\ :\ \eta_{uu}=0, \tag{3.18}$$

$$u_{xx}\ :\ \eta_{ux}-\xi_{xx}=0, \tag{3.19}$$

$$u_t\ :\ 3\xi_x-\tau_t=0, \tag{3.20}$$

$$u_x\ :\ 6\eta+12u\xi_x-\xi_t+3\eta_{uxx}-\xi_{xxx}=0, \tag{3.21}$$

$$\textbf{Rest}\ :\ \eta_{xxx}+6u\eta_x+\eta_t=0 \tag{3.22}$$

Equations (3.15) and (3.16) imply that $\tau=\tau(t)$.

By rewriting equation (3.20), we obtain $\xi_x=\frac{\tau_t(t)}{3}$ and hence

$$\xi=\frac{\tau_t(t)}{3}x+a(t,u), \tag{3.23}$$

for some arbitrary function a of t and x.

Since $\xi_u=0$ as in equation (3.17), we have that

$$\xi = \frac{\tau_t(t)}{3}x + a(t) \tag{3.24}$$

Integration of equation (3.18) twice with respect to u yields

$$\eta = b(t,x)u + c(t,x), \tag{3.25}$$

where b and c are arbitrary functions of t and x.
From the equations (3.19) and (3.24), we deduce that $\eta_{ux} = b_x(t, x) = \xi_{xx} = 0$, which confirms that $b = b(t)$ and hence

$$\eta = b(t)u + c(t,x) \tag{3.26}$$

We substitute equation (3.26) into equation (3.22) to have

$$c_{xxx}(t,x) + 6uc_x(t,x) + b_t(t)u + c_t(t,x) = 0 \tag{3.27}$$

Splitting equation (3.27) on powers of u yields,

$$u \ : \ 6c_x(t,x) + b_t(t) = 0, \tag{3.28}$$

$$u^0 \ : \ c_{xxx}(t,x) + c_t(t,x) = 0 \tag{3.29}$$

Rewriting equation (3.28) gives $c_x(t,x) = \frac{-b_t(t)}{6}$ which reveals that $c_{xxx}(t, x) = 0$, and as a consequence of equation (3.29), one finds that $c_t(t, x) = 0$, showing that, $c = c(x)$.
So far we have that,

$$\tau = \tau(t), \tag{3.30}$$

$$\xi = \frac{\tau_t(t)}{3}x + a(t), \tag{3.31}$$

$$\eta = b(t)u + c(x) \tag{3.32}$$

Substitution of equations (3.31) and (3.32) into equation (3.21) yields

$$6b(t)u + 6c(x) + 4u\tau_t(t) - \frac{\tau_{tt}(t)}{3}x + a_t(t) = 0, \tag{3.33}$$

which after splitting on powers of u gives,

$$u \ : \ 6b(t) + 4\tau_t(t) = 0, \tag{3.34}$$

$$u^0 \ : \ 6c(x) - \frac{\tau_{tt}(t)}{3}x - a_t(t) = 0, \tag{3.35}$$

From equations (3.34) and (3.35), we get $b(t) = \frac{-2\tau_t(t)}{3}$ and $c(x) = \frac{\tau_{tt}(t)}{18}x + \frac{a_t(t)}{6}$ respectively, so that equation (3.32) be comes,

$$\eta = \frac{-2\tau_t(t)}{3}u + \frac{\tau_{tt}(t)}{18}x + \frac{a_t(t)}{6} \tag{3.36}$$

Substituting the expression for η from equation (3.36) into equation (3.22) produces

$$-\frac{\tau_{tt}(t)}{3}u + \frac{\tau_{ttt}(t)}{18}x + \frac{a_{tt}(t)}{6} = 0 \tag{3.37}$$

We then split equation (3.37) on powers of u and x and integrate the resulting equations with respect to t to discover that $\tau(t) = 3C_1t + C_2$ and $a(t) = 6C_3t + C_4$.
Finally,

$$\tau = 3C_1t + C_2, \tag{3.38}$$

$$\xi = C_1x + 6C_3t + C_4, \tag{3.39}$$

$$\eta = -2C_1u + C_3 \tag{3.40}$$

The lengthy calculations above prove that Korteweg-de Vries equation (1.3) admits a four-dimensional Lie algebra spanned by

$$X_1 = \frac{\partial}{\partial x}, \tag{3.41}$$

$$X_2 = \frac{\partial}{\partial t}, \tag{3.42}$$

$$X_3 = 6t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \tag{3.43}$$

$$X_4 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u} \tag{3.44}$$

Remark 3.1: The first two symmetries represent space and time translations respectively while the third represents Galilean boost and the fourth represents scaling symmetry.

3.2 Commutator table for the Lie point symmetries

We evaluate the commutation relations for the symmetry generators. By definition of Lie bracket in section (2.28), for example, we have that

$$[X_1, X_2] = X_1X_2 - X_2X_1 = \left(\frac{\partial}{\partial x}\frac{\partial}{\partial t}\right) - \left(\frac{\partial}{\partial t}\frac{\partial}{\partial x}\right) = 0 \tag{3.46}$$

Remark 3.2: The remaining commutation relations are obtained analogously. We present all commutation relations in table (1) below.

Table 1: A commutator table for the Lie algebra spanned by the symmetries of Korteweg-de Vries equation

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	0	X_1
X_2	0	0	6 X_1	$3X_2$
X_3	0	-6X_1	0	-2 X_3
X_4	-X_1	-3X_2	2X_3	0

3.3 One-parameter groups of transformations

The corresponding one-parameter group of transformations can be determined by solving the Lie equations as defined in section (2.17). Let $T_\epsilon i$ be the group of transformations for each X_i, i = 1, 2, 3, 4. We display how to obtain $T_\epsilon i$ from X_i by finding one-parameter group for the infinitesimal generator X_1, namely,

$$X_1 = \frac{\partial}{\partial x} \tag{3.47}$$

In particular, we have the Lie equations

$$\begin{aligned} \frac{d\bar{t}}{\epsilon} &= 0, \quad \bar{t}\Big|_{\epsilon=0} = t, \\ \frac{d\bar{x}}{\epsilon} &= 1, \quad \bar{x}\Big|_{\epsilon=0} = x, \\ \frac{d\bar{u}}{\epsilon} &= 0, \quad \bar{u}\Big|_{\epsilon=0} = u. \end{aligned} \tag{3.48}$$

Solving the system (3.48) one obtains,

$$\bar{t} = t, \quad \bar{x} = x + \epsilon, \quad \bar{u} = u, \tag{3.49}$$

and hence the one-parameter group $T_{\epsilon 1}$ corresponding to the operator X_1 is

$$T_{\epsilon_1}: \quad \bar{t} = t, \quad \bar{x} = x + \epsilon_1, \quad \bar{u} = u \tag{3.50}$$

The other three groups are obtained analogously and we get the following one-parameter groups:

$$\begin{aligned} T_{\epsilon_1}: &\quad \bar{t}=t, \quad \bar{x}=x+\epsilon_1, \quad \bar{u}=u,\\ T_{\epsilon_2}: &\quad \bar{t}=t+\epsilon_2, \quad \bar{x}=x, \quad \bar{u}=u,\\ T_{\epsilon_3}: &\quad \bar{t}=t, \quad \bar{x}=x+6\epsilon_3 t, \quad \bar{u}=u+\epsilon_3,\\ T_{\epsilon_4}: &\quad \bar{t}=te^{3\epsilon_4}, \quad \bar{x}=xe^{\epsilon_4}, \quad \bar{u}=ue^{-2\epsilon_4}. \end{aligned} \quad (3.51)$$

3.4 Symmetry transformations

We employ the symmetries obtained in section (3.1) to find exact solutions for Korteweg-de Vries equation. In a more general sense of symmetry analysis, we can either transform known solutions by the groups or construct group-invariant solutions. By the criterion of invariance if $\bar{u} = \Gamma(\bar{t}, \bar{x})$ admits to equation (1.3), then so does

$$\phi(t,x,u,\epsilon) = \Gamma(\varphi_1(t,x,u,\epsilon), \varphi_2(t,x,u,\epsilon)) \quad (3.52)$$

For example the one-parameter family of solutions generated by the group

$$T_{\epsilon_1}: \bar{t}=t, \quad \bar{x}=x+\epsilon_1, \quad \bar{u}=u, \quad (3.53)$$

if $\bar{u} = \Gamma(\bar{t}, \bar{x})$is a solution, then $u = \Gamma(t, x + \epsilon_1)$ is also a solution.

The generated solutions include of all the one-parameter groups include:

$$\begin{aligned} T_{\epsilon_1}: &u^{(1)} = \Gamma(t, x+\epsilon_1),\\ T_{\epsilon_2}: &u^{(2)} = \Gamma(t+\epsilon_2, x),\\ T_{\epsilon_1}: &u^{(3)} = \Gamma(t, x+6t\epsilon_3) - \epsilon_3,\\ T_{\epsilon_1}: &u^{(4)} = e^{2\epsilon_4}\Gamma(te^{3\epsilon_4}, xe^{\epsilon_4}). \end{aligned} \quad (3.54)$$

3.5 Group-invariant solutions of (1.3)

Now we compute the group invariant solutions of Korteweg-de Vries equation.

1) Translationally-invariant solutions

We consider the space translation operator

$$X_1 = \frac{\partial}{\partial x} \quad (3.55)$$

Characteristic equations associated with the operator (3.55) are

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{1} = \frac{\mathrm{d}u}{0}, \quad (3.56)$$

which give two invariants $J_1 = t$ and $J_2 = u$. Therefore, $u = \psi(t)$ is the group-invariant solution for some arbitrary function ψ. Substitution of $u = \psi(t)$ into (1.3) yields

$$\psi'(t) = 0, \quad (3.57)$$

whose solution is

$$\psi(t) = C_1, \quad (3.58)$$

for C_1 an arbitrary constant. Hence the group-invariant solution of (1.3) under the space translation operator (3.55) is

$$u(t,x) = C_1 \quad (3.59)$$

2) Stationary solutions

Consider the time translation operator

$$X_2 = \frac{\partial}{\partial t}. \quad (3.60)$$

The Lagrangian system associated with the operator (3.60) is

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{0} = \frac{\mathrm{d}u}{0}, \quad (3.61)$$

whose invariants are $J_1 = x$ and $J_2 = u$. So, $u = \psi(x)$ is the group-invariant solution. Substituting of $u = \psi(x)$ into (1.3) yields

$$6\psi'(x)\psi(x) + \psi'''(x) = 0 \tag{3.62}$$

All the stationary solutions have the form

$$u = \psi(x), \tag{3.63}$$

for some arbitrary function ψ satisfying the equation (3.62) or equation (3.64)

$$(\psi')^2 = -2\psi^3 + 2k\psi + l, \tag{3.64}$$

obtained from (3.62) by two integrations, for which k and l are constants of integration. The general invariant solution takes the form [18]

$$u = -2\Phi(x), \tag{3.65}$$

where $\Phi(x)$ is the Weierstrass elliptic function satisfying the equation,

$$\Phi(x)'^2 = 4\Phi(x)^3 - g_2\Phi(x) - g \tag{3.66}$$

For some real roots r_1, r_2, r_3, of the cube polynomial on the right hand side of (3.66), the solutions (3.63) could be written in the following forms: [18]

1) If $r_1 < r_2 < r_3$, then $u = u(x)$, is a limited function and

$$u = \frac{2a}{s^2}\mathrm{dn}^2\left(\sqrt{\frac{a}{s^2}}x, s\right) + \gamma, \tag{3.67}$$

a cnoidal wave where $\mathrm{dn}^2(x, s)$ is the Jacobian elliptic function with modulus $s = \sqrt{\frac{(r_3-r_2)}{(r_3-r_1)}}$ wherein $a = \frac{(r_3-r_2)}{2}$ is the amplitude of a wave, and $\gamma = r_1$.

2) If $r_1 = r_2 < r_3$, then $u \to r_1$, u^J, $u^{JJ} \to 0$ when $|x| \to \infty$, and get the solitary wave,

$$u = (r_3 - r_1)\,\mathrm{sech}^2\left(\sqrt{\frac{(r_3 - r_1)}{2}}\right) + r_1 \tag{3.68}$$

3) If $r_1 = r_2 = r_3$, then

$$u = \frac{-2}{(x-c)^2}, \quad x \neq c. \tag{3.69}$$

3) Galilean-invariant solutions.
Consider the Galilean boost operator

$$X_3 = 6t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}. \tag{3.70}$$

Characteristic equations associated to the operator (3.70) yield two invariants $J_1 = t$ and $J_2 = -u + \frac{x}{6t}$. As a result, the group-invariant solution of (1.3) for this case is $J_2 = \varphi(J_1)$, for φ an arbitrary function. That is,

$$u(t, x) = -\phi(t) + \frac{x}{6t}. \tag{3.71}$$

Substitution of the value of u from equation (3.71) into equation (1.3) yields a first order ordinary differential equation $\phi'(t) + \frac{\phi(t)}{t} = 0$ whose general solution is $\phi(t) = \frac{\delta}{t}$ with δ an arbitrary constant of integration. Hence, the group-invariant solution under X_3 is

$$u(t, x) = \left(\frac{x + A}{6t}\right), \tag{3.72}$$

where $A = 6\delta$ and t /= 0.

4) Scale-invariant solutions

Last but not least, we consider the scaling operator

$$X_4 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} \quad (3.73)$$

The associated Lagrangian equations to (3.73) yield two invariants, $J_1 = \frac{x^3}{t}$ and $J_2 = ux^2$. Thus, the group-invariant solution is u(t, x) = ϕ(λ) , λ = x3 . Generally, scale-invariant solutions take the form

$$u = x^{-2}\varphi\left(\frac{x^3}{t}\right), \quad t \neq 0, \quad (3.74)$$

where for the variable $\lambda = \frac{x^3}{t}$, and ϕ must satisfy equation (3.75)

$$24\lambda\varphi' - 24\varphi + 27\lambda^3\varphi''' - \lambda^2\varphi' - 12\varphi^2 + 18\lambda\varphi\varphi' = 0. \quad (3.75)$$

3.6 Travelling wave solutions

We obtain travelling wave solutions of Korteweg-de Vries equation by considering a linear com- bination of the symmetries X_1 and X_2, namely, [18]

$$X = c\frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \quad \text{for some constant } c. \quad (3.76)$$

The characteristic equations are

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{c} = \frac{\mathrm{d}u}{0} \quad (3.77)$$

We get two invariants, $J_1 = u$ and $J_2 = x - ct$. So the group-invariant solution is

$$u(t,x) = \varphi(x - ct), \quad (3.78)$$

for some arbitrary function ϕ and c the velocity of the wave.

Substitution of u into (1.3) yields a third order ordinary differential equation

$$-c\varphi' + 6\varphi\varphi' + \varphi''' = 0. \quad (3.79)$$

Integration of equation (3.79) with respect to ϕ yields

$$-c\varphi + 3\varphi^2 + \varphi'' = 0, \quad (3.80)$$

where we take 0 as a constant of integration. The second integration is done after multiplying equation (3.80) by $2\phi^J$ and we get

$$(\varphi')^2 = c\varphi^2 - 2\varphi^3, \quad (3.81)$$

Or

$$\frac{\mathrm{d}\varphi}{\sqrt{c\varphi^2 - 2\varphi^3}} = \mathrm{d}\xi, \quad (3.82)$$

where $\xi = x - ct$.

By the change of variable $\phi = {}^c$ sech$^2(\xi)$, we get a one-soliton solution,

$$\varphi(x,t) = \frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x - ct)\right) \quad (3.83)$$

4. Conservation laws

We derive conservation laws for Korteweg-de Vries equation by using the multiplier method.

4.1 The multipliers

We make use of the Euler-Lagrange operator defined as defined in (2.29) to look for a zeroth order multiplier $\Lambda = \Lambda(t, x, u)$. The resulting determining equation for computing Λ is

$$\frac{\delta}{\delta u}\left[\Lambda\{u_t + 6uu_x + u_{xxx}\}\right] = 0 \tag{4.1}$$

Expansion of equation (4.1) yields

$$\Lambda_u(u_t + 6uu_x + u_{xxx}) + 6u_x\Lambda - D_t(\Lambda) - 6D_x(\Lambda u) - D_x^3(\Lambda) = 0. \tag{4.2}$$

Invoking the total derivatives defined in (3.8) and (3.9) on equation (4.2) results in

$$\Lambda_t + 6u\Lambda_x + \Lambda_{xxx} + (3\Lambda_{xxu})u_x + (3\Lambda_{xuu})u_x^2 + \Lambda_{uuu}u_x^3 + 3\Lambda_{xu}u_{xx} + (3\Lambda_{uu})u_xu_{xx} = 0 \tag{4.3}$$

Splitting equation (4.3) on derivatives of *u* produces an overdetermined system of six partial dif- ferential equations, namely,

$$u_x : \Lambda_{xxu} = 0, \tag{4.4}$$

$$u_x^2 : \Lambda_{xuu} = 0, \tag{4.5}$$

$$u_x^3 : \Lambda_{uuu} = 0, \tag{4.6}$$

$$u_{xx} : \Lambda_{xu} = 0, \tag{4.7}$$

$$u_xu_{xx} : \Lambda_{uu} = 0, \tag{4.8}$$

$$rest : \Lambda_t + 6u\Lambda_x + \Lambda_{xxx} = 0. \tag{4.9}$$

Note that equations (4.4) and (4.5) are trivially satisfied by equation (4.7). Likewise equation (4.8) admits equation (4.6). Integrating equation (4.8) twice with respect to *u* gives

$$\Lambda = \mathfrak{a}(t,x)u + \mathfrak{b}(t,x), \tag{4.10}$$

for some arbitrary functions *a* and *b* of *t* and *x*. Substitution of the value of Λ from (4.10) into equation (4.7) yields,

$$\mathfrak{a}_x(t,x) = 0, \tag{4.11}$$

from which we get that $a = a(t)$, and consequently

$$\Lambda = \mathfrak{a}(t)u + \mathfrak{b}(t,x). \tag{4.12}$$

By substituting the value of Λ from equation (4.12) into equation (4.9), we have

$$\mathfrak{a}_t(t,x)u + \mathfrak{b}_t(t,x) + 6u\mathfrak{b}_x(t,x) + \mathfrak{b}_{xxx}(t,x) = 0, \tag{4.13}$$

which splits on powers of *u* as

$$u \;:\; \mathfrak{a}_t(t,x) + 6\mathfrak{b}_x(t,x) = 0, \tag{4.14}$$

$$u^0 \;:\; \mathfrak{b}_t(t,x) + \mathfrak{b}_{xxx}(t,x) = 0. \tag{4.15}$$

Equation (4.14) implies that

$$\mathfrak{b}_x(t,x) = \frac{-\mathfrak{a}_t(t,x)}{6}, \tag{4.16}$$

from which $b_{xxx}(t, x) = 0$. Thus by equation (4.15), we deduce that $b_t(t, x) = 0$, which in turn verifies that

$$\mathfrak{b} = \mathfrak{b}(x). \tag{4.17}$$

Upon integration of (4.16) with respect x, we get

$$\mathfrak{b}(t,x) = \frac{-\mathfrak{a}_t(t,x)}{6}x + \mathfrak{c}(t), \tag{4.18}$$

for some arbitary function c. Since $b = b(x)$ as implied by equation (4.17) and $a = a(t)$ as shown in equation (4.11), we must have that $c = C_1$ and $a_t(t, x) = C_2$ for some constants C_1 and C_2.
This gives $a(t, x) = C_2t + C_3$ and $\mathfrak{b}(t,x) = C_1 + \frac{-C_2}{6}x$ for C_3 a constant of integration.

Hence we have,

$$\Lambda = C_1 + C_2\left(tu - \frac{x}{6}\right) + C_3 u, \tag{4.19}$$

which is a linear combination three nontrivial conservation law multipliers

$$\Lambda^1(t,x,u) = 1, \quad \Lambda^2(t,x,u) = tu - \frac{x}{6} \quad \text{and} \quad \Lambda^3(t,x,u) = u. \tag{4.20}$$

Remark 4.1. Recall from (2.35) that a multiplier Λ for equation(1.3) has the property that for the density $T^t = T^t(t, x, u, u_x)$ and flux $T^x = T^x(t, x, u, u_x, u_{xx})$,

$$\Lambda\left(u_t + 6uu_x + u_{xxx}\right) = D_t T^t + D_x T^x. \tag{4.21}$$

We derive conservation law corresponding to each of the multipliers.'

1) Conservation law for the multiplier $\Lambda^1(t, x, u) = 1$.
Expansion of equation (4.21) gives

$$u_t + 6uu_x + u_{xxx} = T^t_t + u_t T^t_u + u_{tx} T^t_{u_x} + T^x_x + u_x T^x_u + u_{xx} T^x_{u_x} + u_{xxx} T^x_{u_{xx}}. \tag{4.22}$$

Splitting equation (4.22) on third derivatives of u yields

$$u_{xxx} \; : \; T^x_{u_{xx}} = 1, \tag{4.23}$$

$$\text{Rest} \; : \; u_t + 6uu_x = T^t_t + T^t_u u_t + T^t_{u_x} u_{tx} + T^x_x + T^x_u u_x + T^x_{u_x} u_{xx}. \tag{4.24}$$

By integrating equation (4.23) with respect to u_{xx}, we deduce that

$$T^x = u_{xx} + A(t,x,u,u_x), \tag{4.25}$$

for A an arbitrary function of its arguments.
Substituting the expression of T^x from (4.25) into equation (4.24) we get

$$u_t + 6uu_x = T^t_t + T^t_u u_x t + T^t_{u_x} u_{tx} + A_x + A_u u_x + A_{u_x} u_{xx}, \tag{4.26}$$

which splits on second derivatives of u, to give

$$u_{tx} \; : \; T^t_{u_x} = 0, \tag{4.27}$$

$$u_{xx} \; : \; A_{u_x} = 0, \tag{4.28}$$

$$\text{Rest} \; : \; u_t + 6uu_x = T^t_t + T^t_u u_t + A_x + A_u u_x. \tag{4.29}$$

Integrating equations (4.27) and (4.28) with respect to u_x manifests that

$$T^t = B(t,x,u), \tag{4.30}$$

$$A = A(t,x,u), \tag{4.31}$$

for B an arbitrary function of t, x and u.
Substituting the expressions of T^t and A from (4.30) and (4.31) respectively, into equation (4.29) gives

$$u_t + 6uu_x = B_t + B_u u_t + A_x + A_u u_x, \tag{4.32}$$

which splits on the derivatives of u to yield

$$u_t \quad : \quad B_u = 1, \tag{4.33}$$

$$u_x \quad : \quad A_u = 6u, \tag{4.34}$$

$$\text{Rest} \quad : \quad A_x + B_t = 0. \tag{4.35}$$

By integrating equations (4.33) and (4.34) with respect to u, we find that

$$B = u + C(t, x), \tag{4.36}$$

$$A = 3u^2 + D(t, x) \tag{4.37}$$

for C and D arbitrary function of the arguments t and x.
Substitution of the expressions of A and B into equation (4.35) shows that $D_x + C_t = 0$. Since C and D contribute to the trivial part of the conservation law, we take $C = D = 0$. We get the conserved vectors

$$T^t = u, \tag{4.38}$$

$$T^x = 3u^2 + u_{xx}, \tag{4.39}$$

from which the conservation law corresponding to the multiplier $\Lambda_1 = 1$ is given by

$$D_t(u) + D_x(3u^2 + u_{xx}) = 0. \tag{4.40}$$

Remark 4.2. The fact that $\Lambda_1 = 1$ is multiplier is a sufficient evidence that Korteweg-de Vries equation (1.3) is itself a conservation law.

2) Conservation law for the multiplier $\Lambda^2(t, x, u) = tu - \frac{x}{6}$.
Substituting Λ^2 into equation (4.21) and expanding yields

$$u_t\left(tu - \frac{x}{6}\right) + u_x(6tu^2 - xu) + u_{xxx}\left(tu - \frac{x}{6}\right) = T^t_t + T^t_u u_t + T^t_{u_x} u_{tx} + T^x_x + T^x_u u_x + T^x_{u_x} + u_{xx} + T^x_{u_{xx}} u_{xxx}. \tag{4.41}$$

Splitting equation (4.41) on third derivatives of u gives

$$u_{xxx} : T^x_{u_{xx}} = \left(tu - \frac{x}{6}\right), \tag{4.42}$$

$$\text{Rest} : u_t\left(tu - \frac{x}{6}\right) + u_x\left(6tu^2 - xu\right) = T^t_t + T^t_u u_t + T^t_{u_x} u_{tx} + T^x_x + T^x_u u_x + T^x_{u_x} u_{xx}. \tag{4.43}$$

Integration of equation (4.42) with respect to u_{xx} reveals that

$$T^x = \left(tu - \frac{x}{6}\right) u_{xx} + A(t, x, u, u_x), \tag{4.44}$$

for A an arbitrary function of its arguments.
Substituting the expression of T^x from (4.44) into equation (4.43) we have

$$u_t\left(tu - \frac{x}{6}\right) + u_x\left(6tu^2 - xu\right) = [A_u + tu_{xx}]u_x + A_{u_x} u_{xx} + T^t_t + T^t_u u_t + T^t_{u_x} u_{tx} \tag{4.45}$$

Splitting the equation (4.46) on second derivatives of u gives

$$u_{tx} \quad : \quad T^t_{u_x} = 0, \tag{4.47}$$

$$u_{xx} \;:\; A_{u_x} = \frac{1}{6} - tu_x, \tag{4.48}$$

$$\text{Rest} \;:\; u_t\left(tu - \frac{x}{6}\right) + u_x\left(6tu^2 - xu\right) = T^t_t + T^t_u u_t + A_x + A_u u_x \tag{4.49}$$

Integration of equations (4.47) and (4.48) with respect to u_x yields

$$T^t = \alpha(t, x, u), \tag{4.50}$$

$$A = \frac{u_x}{6} - \frac{u_x^2 t}{2} + \delta(t, x, u), \tag{4.51}$$

where α and δ are arbitrary functions of t, x and u. Substituting the values of T^t and A from (4.50) and (4.51) respectively into equation (4.49) gives

$$u_t\left(tu - \frac{x}{6}\right) + u_x\left(6tu^2 - xu\right) = A_t + A_u u_t + \delta_x + \delta_u u_x \tag{4.52}$$

The equation (4.52) splits on first derivatives of u to give

$$u_t \;:\; A_u = tu - \frac{x}{6}, \tag{4.53}$$

$$u_x \;:\; \delta_u = 6u^2 t - xu, \tag{4.54}$$

$$\text{Rest} \;:\; T_t + \delta_x = 0. \tag{4.55}$$

By integrating equations (4.53) and (4.54) with respect to u , we have that

$$A = \frac{tu^2}{2} - \frac{xu}{6} + \gamma(t, x), \tag{4.56}$$

$$\delta = 2u^3 t - \frac{xu^2}{2} + \rho(t, x), \tag{4.57}$$

where ρ and γ are arbitrary functions of their arguments. Substituting the values of A and δ into equation (4.55) gives $\gamma_t + \rho_x = 0$. We take note that γ and ρ contribute to the trivial part of the conservation law thus we take them to be zero to give the conserved vectors

$$T^t = \frac{tu^2}{2} - \frac{xu}{6}, \tag{4.58}$$

$$T^x = 2u^3 t - \frac{xu^2}{2} + \frac{u_x}{6} - \frac{u_x^2 t}{2} + \left(tu - \frac{x}{6}\right) u_{xx} \tag{4.59}$$

Hence the conservation law corresponding to the multiplier $\Lambda_2 = tu - \underline{x}$ is given by

$$D_t\left(\frac{tu^2}{2} - \frac{xu}{6}\right) + D_x\left(2u^3 t - \frac{xu^2}{2} + \frac{u_x}{6} - \frac{u_x^2 t}{2} + \left(tu - \frac{x}{6}\right) u_{xx}\right) = 0 \tag{4.60}$$

3) Conservation law for the multiplier $\Lambda^3(t, x, u) = u$.
Substituting Λ^3 into equation (4.21) and expanding, the result is

$$u_t u + 6u^2 u_x + u u_{xxx} = T^t_t + u_t T^t_u + u_{tx} T^t_{u_x} + T^x_x + u_x T^x_u + u_{xx} T^x_{u_x} + u_{xxx} T^x_{u_{xx}} \tag{4.61}$$

We then split equation (4.61) on third derivatives of u, to obtain

$$u_{xxx} \;:\; T^x_{u_{xx}} = u \tag{4.62}$$

$$\text{Rest} \;:\; uu_t + 6u^2 u_x = T^t_t + T^t_u u_t + T^t_{u_x} u_{tx} + T^x_x + T^x_u u_x + T^x_{u_x} u_{xx}. \tag{4.63}$$

Integration of equation (4.62) with respect to u_{xx} yields

$$T^x = uu_{xx} + A(t, x, u, u_x), \tag{4.64}$$

where A is an arbitrary function of its arguments.
Substituting the expression of T^x from (4.64) into equation (4.63) we have

$$uu_t + 6u^2u_x = T^t_t + T^t_u u_t + T^t_{u_x} u_{tx} + A_x + [A_u + u_{xx}]u_x + A_{u_x} u_{xx} \tag{4.65}$$

By splitting the above equation (4.65) on second derivatives of u, we find

$$u_{tx} : T^t_{u_x} = 0, \tag{4.66}$$

$$u_{xx} : A_{u_x} = -u_x, \tag{4.67}$$

$$\text{Rest} : uu_t + 6u^2u_x = T^t_t + T^t_u u_t + A_x + A_u u_x \tag{4.68}$$

By integrating equations (4.66) and (4.67) with respect to u_x, one finds that

$$T^t = \theta(t, x, u), \tag{4.69}$$

$$A = \frac{-u_x^2}{2} + k(t, x, u), \tag{4.70}$$

for arbitrary functions θ and k.
Substitution of the expressions of T^t and A from equations (4.69) and (4.70) respectively into equation (4.68), we get

$$uu_t + 6u^2u_x = \theta_t + \theta_u u_t +_x + k_x + k_u u_x, \tag{4.71}$$

which splits on second derivatives of u as,

$$u_t : \theta_u = u, \tag{4.72}$$

$$u_x : k_u = 6u^2, \tag{4.73}$$

$$\text{Rest} : \theta_t + A_x = 0 \tag{4.74}$$

We integrate equations (4.72) and (4.73) with respect to u and obtain

$$\theta = \frac{u^2}{2} + c(t, x), \tag{4.75}$$

$$k = 2u^3 + a(t, x), \tag{4.76}$$

for some arbitrary functions c and a of t and x.
Lastly, we substitute the values of θ and A into equation (4.74) which gives $\theta_t + A_x = 0$. We take note that a and c contribute to the trivial part of the conservation law thus we take them to be zero and get the conserved vectors

$$T^t = \frac{u^2}{2}, \tag{4.77}$$

$$T^x = 2u^3 + uu_{xx} - \frac{1}{2}u_x^2. \tag{4.78}$$

Hence the conservation law corresponding to the multiplier $\Lambda_3 = u$ is given by

$$D_t\left(\frac{u^2}{2}\right) + D_x\left(2u^3 + uu_{xx} - \frac{1}{2}u_x^2\right) = 0. \tag{4.79}$$

4. Conclusion

In this paper, we have employed symmetry analysis to study a kdv type equation. A four-dimensional Lie algebra of symmetries was found for the nonlinear KdV equation.Our Lie algebra is generated by space and time translations, Galilean boost and scaling symmetries where the scaling symme- try acts on three variables. Associated to each symmetry, we obtained symmetry reductions that gave six nontrivial solutions for the kdv equation. All the solutions describe the various states of any system that can be modeled by a kdv type equation. The obtained solutions can be used as a benchmark against numerical simulations. In future, we will construct more conservation laws by Ibragimov approach and generalize a study of such problems.

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6. Author's contribution

The author contributed wholly in writing this article and declares no conflict of interest.

7. References

1. Almeida M, Magalhaes M, Moreira I. Lie symmetries and invariants of the lotka- volterra system. Journal of Mathematical Physics. 1995; 36(4):1854-1867.
2. Arrigo DJ. Symmetry analysis of differential equations: an introduction. John Wiley & Sons, 2015.
3. Bluman G, Anco S. Symmetry and integration methods for differential equations, volume 154. Springer Science & Business Media, 2008.
4. Bluman GW, Kumei S. Symmetries and differential equations, volume 81. Springer Science & Business Media, 1989.
5. Bluman GW, Cheviakov AF, Anco SC. Applications of symmetry methods to partial differential equations, volume 168. Springer, 2010.
6. Brauer F, Castillo-Chavez C, Castillo-Chavez C. Mathematical models in population biology and epidemiology, volume 2. Springer, 2012.
7. Ibragimov N. Elementary Lie group analysis and ordinary differential equations. Wiley, 1999.
8. Ibragimov NH. CRC handbook of Lie group analysis of differential equations, volume 1-3. CRC press, 1994.
9. Ibragimov NH. Selected works. Volume 1-4. ALGA publications, Blekinge Institute of Technology, 2006-2009. Ibragimov NH. A new conservation theorem. Journal of Mathematical Analysis and Applications. 2007; 333(1):311-328.
10. Ibragimov NH. A Practical Course in Differential Equations and Mathematical Modelling: Classical and New Methods. Nonlinear Mathematical Models. Symmetry and Invariance Principles. World Scientific Publishing Company, 2009.
11. Khalique CM, Abdallah SA. Coupled Burgers equations governing polydispersive sedimentation: A lie symmetry approach. Results in Physics. 2020; 16.
12. LeVeque RJ. Numerical methods for conservation laws, volume 3. Springer, 1992.
13. Lie S. Vorlesungen u¨ber Differentialgleichungen mit bekannten infinitesimalen Transforma- tionen. BG Teubner, 1891.
14. Mhlanga I, Khalique C. Travelling wave solutions and conservation laws of the Korteweg-de Vries-Burgers Equation with Power Law Nonlinearity. Malaysian Journal of Mathematical Sciences. 2017; 11:1-8.
15. Murray JD. An introduction. Springer, 2002.
16. Noether E. Invariant variations problem. Nachr. Konig. Gissel. Wissen, Gottingen. Math. Phys. Kl, 1918, 235-257.
17. Olver PJ. Applications of Lie groups to differential equations, volume 107. Springer Science & Business Media, 1993.
18. Ovsyannikov L. Lectures on the theory of group properties of differential equations. World Scientific Publishing Company, 2013.
19. Owino JO. Exact symmetry reduction solutions of a nonlinear coupled system of korteweg-de vries equations. International Journal of Advanced Multidisciplinary Research and Studies. 2022; 2:76-87.
20. Owino JO, Okelo B. Lie group analysis of a nonlinear coupled system of korteweg-de vries equations. European Journal of Mathematical Analysis. 2021; 1:133-150.
21. Owuor J. Conserved quantities of a nonlinear coupled system of korteweg-de vries equations. International Journal of Mathematics and Computer Research. 2022; 10(5):2673-2681.
22. Owuor J. Exact symmetry reduction solutions of a nonlinear coupled system of korteweg-de vries equations. International Journal of Advanced Multidisciplinary Research and Studies. 2022; 2(3):76-87.
23. PE H. Symmetry methods for differential equations: a beginner's guide, volume 22. Cam- bridge University Press, 2000.
24. Wazwaz AM. Partial differential equations and solitary waves theory. Springer Science & Business Media, 2010.