Int. j. adv. multidisc. res. stud. 2022; 2(2):494-500

Received: 08-03-2022
Accepted: 18-04-2022

International Journal of Advanced Multidisciplinary Research and Studies

ISSN: 2583-049X

Some approximation estimates for summation integral type operators

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Abstract
In operator theory q-calculus is an active area of research in last few years. Several new q-operators were introduced and their approximation behavior was discussed. In the present paper, we study the Stancu type generalization of Beta-Szasz type operators for their q-analogues. We obtain moments and convergence results in terms of higher order modulus of continuity.

Keywords: q-Integers, q-Beta-Szasz Type Operators, q-Exponential Function, Modulus of Continuity, Weighted Approximation

AMS Subject Classification: 41A25, 41A35

1. Introduction

Many researchers introduced several q-operators and discussed their approximation properties. Very first in the year 1987, A. Lupas [16] gave the first q-analogue of classical Bernstein Polynomials. After that Phillips [22] introduced another important q-analogue of Bernstein polynomials. Many researchers worked in this direction and proposed various q-operators and studied their different properties e.g. [9, 10, 11, 15, 17, 18, 19, 20] and [21] etc. For q-discrete operators the convergence estimates were also studied by [1, 2] and [3]. Atakut - Buyukyazici [5, 6] studied the Stancu variants of several well-known operators and estimated some direct results. Actually, the Stancu variant is based on two parameters and it generalizes the original operator. Motivated by the recent research on Stancu type operators, we introduced the Stancu type generalization of the Beta-Szasz operators.

For $q \in (0,1)$ and $0 \leq \alpha \leq \beta$, we propose the q-Beta-Szasz-Stancu operators as

(1) $$B_{n,\alpha,\beta}^{q}(f,x) = \sum_{k=0}^{\infty} b_{n,k}^{q}(x) \int_0^{q/1-q^n} q^{-k-1} s_{n,k}^{q}(t) f\left(\frac{[n]_q t q^{-k-1}+\alpha}{[n]_q+\beta}\right) d_q t,$$

where $b_{n,k}(x)$ and $s_{n,k}(t)$ are Beta and Szasz basis functions defined as

$$b_{n,k}^{q}(x) = \frac{q^{k(k-1)/2}}{B_q(k+1,n)} \frac{x^k}{(1+x)_q^{n+k+1}}.$$

And $$s_{n,k}^{q}(t) = E_q(-[n]_q t) \frac{([n]_q t)^k}{[k]_q!}.$$

As a special case when $\alpha = \beta = 0$ and q = 1, the above operators reduce to the Beta-Szasz operators introduced by Gupta and Srivastava [12]. Aral, Gupta and Agrawal [4] published a book which contains many important results on applications of q-Calculus. For the study on this paper, some notations of q-calculus are described below.

$$[n]_q! = \begin{cases} [n]_q [n-1]_q \dots [1]_q, n = 1.2, \dots \\ 1, n = 0 \end{cases}$$

$$[n]_q = \frac{1-q^n}{1-q}$$

$$(1+x)_q^n = \begin{cases} (1+x)(1+qx)\ldots(1+q^{n-1}x), n = 1.2, \ldots \\ 1, n = 0 \end{cases}$$

According to [14], there are two q-analogues of exponential function e^z

$$e_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{[k]!} = \frac{1}{(1-(1-q)z)_q^{\infty}}, \qquad |z| < \frac{1}{1-q}, \qquad |q| < 1$$

And

$$E_q(z) = \prod_{j=0}^{\infty}(1+(1-q)q^i z) = \sum_{k=0}^{\infty} q^{k(k-1)/2}\frac{z^k}{[k]!} = (1+(1-q)z)_q^{\infty}, \quad |q| < 1$$

Where $(1-x)_q^{\infty} = \prod_{j=0}^{\infty}(1-q^j)x)$

The q-Jackson integrals and q-improper integrals are defined as

$$\int_0^a f(t)d_q t = a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n, \quad a > 0$$

$$\int_0^{\infty/A} f(t)d_q t = (1-q)\sum_{n=-\infty}^{\infty} f(\frac{q^n}{A},)\frac{q^n}{A}, \quad A > 0$$

The two q-Gamma functions are defined as

$$\Gamma_q(x) = \int_0^{\frac{1}{1-q}} t^{x-1}E_q(-qt)d_q t,$$

$$\gamma_q^A(x) = \int_0^{\infty} A(1-q)t^{x-1}e_q(-t)d_q t,$$

For every A, x>0, we can get

$$\gamma_q^{(x)} = K(A,x)\gamma_q^A(x)$$

$$K(A,x) = \frac{1}{(1+A)A^x\left(1+(\frac{1}{A})_q^x\right)\left(1+A_q^{1-x}\right)}.$$

For any n>0

$$K(A,n) = q^{n(n-1)/2}$$

and

$$\Gamma_q(n) = q^{n(n-1)}\gamma_q^A(n).$$

In the present paper, we obtain the moments of the q-Beta-Szasz-Stancu operators and estallish some direct results which include the error estimation in terms of modulus of continuity and the weighted approximation for above said operators.

2. Moment Estimations

This section deals with certain lemmas

Lemma 1: [13] For above operator, for $\alpha=\beta=0$ and 0<q<1, following equalities hold

(1) $B_n^q(1,x)=1$

(2) $B_n^q(t,x)=x\left(1+\frac{1}{q[n]_q}\right)+\frac{1}{[n]_q}$

(3) $B_n^q(t^2,x)=\frac{[n+1]_q[n+2]_q}{q^3[n]_q^2}x^2+\frac{[n+1]_q}{q^2[n]_q^2}(1+2q+q^2)x+\frac{[2]_q}{[n]_q^2}$

Lemma 2: For $q\in(0,1)$ and $0\le\alpha\le\beta$, we have

$$B_{n,\alpha,\beta}^q(1,x)=1$$

$$B_{n,\alpha,\beta}^q(t,x)=\frac{x\left(q[n]_q+1\right)+q(1+\alpha)}{q\left([n]_q+\beta\right)}$$

$$B_{n,\alpha,\beta}^q(t^2,x)=\frac{[n+1]_q[n+2]_qx^2+\left(q[n+1]_q+(1+2q+q^2)+2\alpha([n]_qq^3+q^2)\right)x+([2]_q+\alpha^2+1)q^3}{\left([n]_q+\beta\right)^2q^3}.$$

Proof: By Lemma 1, it is clear that

$$B_{n,\alpha,\beta}^q(1,x)=1$$

Further, we have

$$B_{n,\alpha,\beta}^q(t,x)=\sum_{k=0}^{\infty}b_{n,k}^q(x)\int_0^{\frac{q}{1-q^n}}q^{-k-1}s_{n,k}^q(t)f\left(\frac{[n]_qtq^{-k-1}+\alpha)}{[n]_q+\beta}\right)d_qt$$

$$=\frac{[n]_q}{[n]_q+\beta}B_n^q(t,x)+\frac{\alpha}{[n]_q+\beta}B_n^q(1,x)$$

$$=\frac{[n]_q}{[n]_q+\beta}\left(x\left(1+\frac{1}{q[n]_q}\right)+\frac{1}{[n]_q}\right)+\frac{\alpha}{[n]_q+\beta}=\frac{x\left(q[n]_q+1\right)+q(1+\alpha)}{q\left([n]_q+\beta\right)}.$$

We have

$$B_{n,\alpha,\beta}^q(t^2,x)=\sum_{k=0}^{\infty}b_{n,k}^q(x)\int_0^{\frac{q}{1-q^n}}q^{-k-1}s_{n,k}^q(t)f\left(\frac{[n]_qtq^{-k-1}+\alpha)}{[n]_q+\beta}\right)^2d_qt$$

$$=\left(\frac{[n]_q}{[n]_q+\beta}\right)^2B_n^q(t^2,x)+\left(\frac{2[n]_q\alpha}{\left([n]_q+\beta\right)^2}\right)P_n^q(t,x)+\left(\frac{\alpha}{[n]_q+\beta}\right)^2B_n^q(1,x)$$

$$=\left(\frac{[n]_q}{[n]_q+\beta}\right)^2\left(\frac{[n+1]_q[n+2]_q}{q^3[n]_q^2}x^2+\frac{[n+1]_q}{q^2[n]_q^2}(1+2q+q^2)x+\frac{[2]_q}{[n]_q^2}\right)$$

$$+\left(\frac{2[n]_q\alpha}{\left([n]_q+\beta\right)^2}\right)\left(x\left(1+\frac{1}{q[n]_q}\right)+\frac{1}{[n]_q}\right)+\left(\frac{\alpha}{[n]_q+\beta}\right)^2$$

$$=\frac{[n+1]_q[n+2]_qx^2+\left(q[n+1]_q+(1+2q+q^2)+2\alpha([n]_qq^3+q^2)\right)x+([2]_q+\alpha^2+1)q^3}{\left([n]_q+\beta\right)^2q^3}.$$

Lemma 3: For $x \in [0, \infty)$ and $q \in (0,1)$, we obtain the central moments as follows

$$B^q_{n,\alpha,\beta}(t-x,x) = \frac{x(1-q\beta)+q(1+\alpha)}{q([n]_q+\beta)},$$

$$B^q_{n,\alpha,\beta}((t-x)^2,x) = x^2\left[\frac{[n+1]_q[n+2]_q}{([n]_q+\beta)^2q^3} - \frac{x(q[n]_q+1)}{q([n]_q+\beta)} + 1\right]$$

$$+x\left[\frac{[n+1]_q+(1+2q+q^2)}{([n]_q+\beta)^2q^2} + \frac{2\alpha(q[n]_q+1)}{q([n]_q+\beta)^2} - \frac{2(1+\alpha)}{[n]_q+\beta}\right] + \frac{[2]_q+\alpha^2+2\alpha}{([n]_q+\beta)^2}.$$

3. Convergence Estimates

Definition 1. By $C_B[0,\infty)$ we denote the space of real valued continuous bounded functions f on the interval $[0,\infty)$, the norm $\|\cdot\|$ on the space $C_B[0,\infty)$ is given by

$$\|f\| = \sup_{0\le x<\infty}|f(x)|.$$

Definition 2: The Peetre's K-functional is defined by

$$K_2(f,\delta) = inf\{\|f-g\| + \delta\|g''\|: g \in W^2_\infty\},$$

Where $W^2_\infty = \{g \in C_B[0,\infty): g', g'' \in C_B[0,\infty)\}$. Following [8], there exists a positive constant M>0 such that $K_2(f,\delta) \le M\omega_2(f,\sqrt{\delta}),\ \delta > 0$ where the second order modulus of smoothness is given by

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\le\sqrt{\delta}}\ \sup_{0<x\le\infty}|f(x+2h) - 2f(x+h) + f(x)|.$$

Definition 3: For $f \in C_B[0,\infty)$ the usual modulus of continuity is given by

$$\omega_2(f,\delta) = \sup_{0<h\le\delta}\ \sup_{0<x\le\infty}|f(x+h) - f(x)|.$$

Theorem 1: Let $f \in C_B[0,\infty)$ and $q \in (0,1)$, then for all $x \in [0,\infty)$ and $n \in N$; there exists an absolute constant M > 0 such that

$$\left|B^q_{n,\alpha,\beta}(f,x) - f(x)\right| \le M\omega_2(f,\delta_n(x)) + \omega\left(f, \frac{x(1-q\beta)+q(1+\alpha)}{q([n]_q+\beta)}\right).$$

Where $\delta^2_n(x) = \left[B^q_{n,\alpha,\beta}((t-x)^2,x) + \left(\frac{x(1-q\beta)+q(1+\alpha)}{q([n]_q+\beta)}\right)^2\right]^{1/2}.$

Proof: Introducing the auxiliary operators $\overline{B}^q_{n,\alpha,\beta}$ as

(1) $\overline{B}^q_{n,\alpha,\beta}(f,x) = B^q_{n,\alpha,\beta}(f,x) - f\left(x + \frac{x(1-q\beta)+q(1+\alpha)}{q([n]_q+\beta)}\right) + f(x)$, $x \in [0,\infty)$. The operators $\overline{B}^q_{n,\alpha,\beta}(f,x)$ are linear and preserve the linear functions:

(2) $\overline{B}^q_{n,\alpha,\beta}(t-x,x) = 0$

Let $g \in W^2$. Using Taylor's expansion

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-x)g''(u)du,\quad t \in [0,\infty)$$

and (3), we get

$$\overline{B}^{q}_{n,\alpha,\beta}(g,x) = g(x) + \overline{B}^{q}_{n} - f\left(\int_x^t (t-u)g''(u)du, x\right).$$

Hence by (2), we have

$$\left|\overline{B}^{q}_{n,\alpha,\beta}(g,x) - g(x)\right| \le \left|B^{q}_{n,\alpha,\beta}\left(\int_x^t (t-x)g''(u)du,\ x\right)\right| + \left|\int_x^{\frac{x(q[n]_q+1)+q(1+\alpha)}{q([n]_q+\beta)}} \left(\frac{x(q[n]_q+1)+q(1+\alpha)}{q([n]_q+\beta)} - u\right) g''(u)du\right|$$

$$\le \left|B^{q}_{n,\alpha,\beta}\left(\int_x^t |t-u||g''(u)|du,\ x\right)\right| + \int_x^{\frac{x(q[n]_q+1)+q(1+\alpha)}{q([n]_q+\beta)}} \left|\frac{x(q[n]_q+1)+q(1+\alpha)}{q([n]_q+\beta)} - u\right| |g''(u)|du$$

(4) $\le B^{q}_{n,\alpha,\beta}((t-x)^2, x) + \left(\frac{x(1-q\beta)+q(1+\alpha)}{q([n]_q+\beta)}\right)^2 \|g''\| = \delta_n^2(x)\|g''\|.$ and by (2), we have

(5) $\left|\overline{B}^{q}_{n,\alpha,\beta}(f,x)\right| \le \left|B^{q}_{n,\alpha,\beta}(f,x)\right| + 2\|f\| \le 3\|f\|.$

According to results (2), (4) and (5), we get

$$\left|B^{q}_{n,\alpha,\beta}(f,x) - f(x)\right| \le \left|\overline{B}^{q}_{n,}(f-g,x) - (f-g)(x)\right| + \left|\overline{B}^{q}_{n,\alpha,\beta}(g,x) - g(x)\right|$$

$$+\left|f\left(x + \frac{x(1-q\beta)+q(1+\alpha)}{q([n]_q+\beta)}\right) - f(x)\right|$$

$$\le 4\|f-g\|\delta_n^2(x)\|g''\| + \left|f\left(x + \frac{x(1-q\beta)+q(1+\alpha)}{q([n]_q+\beta)}\right) - f(x)\right|.$$

Therefore, taking infimum on the right-hand side over all $g \in W^2$, we get

$$\left|B^{q}_{n,\alpha,\beta}(f,x) - f(x)\right| \le MK_2(f, \delta_n^2(x)) + \omega\left(f, \frac{x(1-q\beta)+q(1+\alpha)}{q([n]_q+\beta)}\right).$$

Using the property of K-functional

$$\left|B^{q}_{n,\alpha,\beta}(f,x) - f(x)\right| \le M\omega_2(f, \delta_n(x)) + \omega\left(f, \frac{x(1-q\beta)+q(1+\alpha)}{q([n]_q+\beta)}\right).$$

This completes the proof of the theorem.

Definition 4: Let $H_{x^2}[0,\infty)$ be the set of all functions f defined on $[0,\infty)$, satisfying the condition $|f(x)| \le K_f(1+x^2)$. Where K_f is a constant depending only on f. By $C_{x^2}[0,\infty)$ we denote the subspace of all continuous functions belonging to $H_{x^2}[0,\infty)$. Also let $C^*_{x^2}[0,\infty)$ be the subspace of all functions $f \in C_{x^2}[0,\infty)$, for which $\log_{x\to\infty} \frac{f(x)}{1+x^2}$ is finite. The norm on $C^*_{x^2}[0,\infty)$

$$\|f\|_{x^2} = \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^2}.$$

We denote the modulus of continuity of f on closed interval [0; a]; a > 0 as by

$$\omega_a(f,\delta) = \sup_{|t-x|\le\delta} \sup_{x,t\in[0,\infty)} |f(t) - f(x).|$$

We observe that for function $f \in C_{x^2}[0,\infty)$, the modulus of continuity $\omega_a(f,\delta)$ tends to zero.

Theorem 2: Let $q = q_n$ satisfies $0 < q_n < 1$ and let $q_n \to 1$ as $n \to \infty$ for each

$f \in C^*_{x^2}[0,\infty)$, we have

$$\lim_{n\to\infty} \left\| B^{q_n}_{n,\alpha,\beta}(f,x) - f(x) \right\|_{x^2} = 0$$

Proof: Following [9], we observe that it is sufficient to verify the following three conditions

(6) $\lim_{n\to\infty} \left\| B^{q_n}_{n,\alpha,\beta}(t^v,x) - x^v \right\|_{x^2} = 0, \quad v = 0,1,2.$

Since $B^{q_n}_{n,\alpha,\beta}(1,x) = 1$ hold for *v*=0

$$\left\| B^{q_n}_{n,\alpha,\beta}(t,x) - x^v \right\|_{x^2} = \sup_{x\in[0,\infty)} \frac{x(1-q\beta) + q(1+\alpha)}{q([n]_q + \beta)} . \frac{1}{1+x^2}.$$

Thus $\left\| B^{q_n}_{n,\alpha,\beta}(t,x) - x \right\|_{x^2} = 0$

$$\left\| B^{q_n}_{n,\alpha,\beta}(t,x) - x^2 \right\|_{x^2} \le \left(\frac{[n+1]_q [n+2]_q}{([n]_q + \beta)^2 q^3} - 1 \right) \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2}$$

$$+ \frac{q[n+1]_q (1 + 2q + q^2) + 2\alpha([n]_q q^3 + q^2)}{([n]_q + \beta)^2 q^3} \sup_{x\in[0,\infty)} \frac{x}{1+x^2}$$

$$+ \left(\frac{[2]_q + \alpha^2 + q^3}{([n]_q + \beta)^2 q^3} \right) \sup_{x\in[0,\infty)} \frac{1}{1+x^2}$$

which implies that $\log_{n\to\infty} \left\| B^{q_n}_{n,\alpha,\beta}(t,x) - x^2 \right\|_{x^2} = 0.$

Hence theorem is proved.

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