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Numerical Radius Power Bounds and Norms of derivations in Banach Algebras

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Abstract
In this paper, we show that $\delta_{P,Q}$ has a lower bound and is utmost equal to the sum of norms of P and Q and also that $\delta_{P,Q}$ is Hermitian and is bounded above by its numerical radius. Finally, we give power bounds for numerical radii of the $\delta_{P,Q}$.

Keywords: Matricial Operator, Hilbert Space, Banach Algebras, Commutators

1. Introduction

Studies on the norm of inner derivations lead [1] to introduce the idea of *S*-universal operators and criteria for the universality for subnormal operators i.e. an operator $T \in B(H)$ such that $\| \delta_T | \tau \| = 2d(T)$, for each norm ideal τ in $B(H)$ and $d(T) = inf_{\lambda \in C}\{\| T - \lambda \|\}$. In [2] it was established the relationship between δ_T, δ_P and $\delta_{T,P}$ on $B(H)$ where the operators T and *P* are S-universal. To be precise; supposing that $T, P \in B(H)$ are S-universal, then $\| \delta_{T,P} | B(H) \| \le \frac{1}{2}(\| \delta_T | B(H) \|) + \| \delta_P | B(H) \|$ and the norm of a generalized derivation implemented by two S-universal operators is less than or equal to half the sum of the norms of inner derivations implemented by each operator [3]. The norm of a derivation δ_T as a mapping of $B(H)$ onto itself is given by $inf \| T - \lambda I \|$ [4]. Kadison, Lance and Ringrose [60] showed that if T is self-adjoint and δ_T maps a subalgebra of $B(H)$ into $B(H)$, then $\| \delta_T \| = inf\{2 \| T - A' \| : A' \in \theta'\}$ where θ' is the commutant of the subalgebra $\theta \subset B(H)$. In [5] the author used an example of a self-adjoint operator to show that the hypothesis that $(\delta(\theta) \subset \theta)$ is inessential, taking θ to be the subalgebra of diagonal matrices with $\theta' = \theta$. Later on, Bonyo [6] investigated the relationship between diameter of the numerical range of an operator $T \in B(H)$ and norms on inner derivations implemented by T on the norm ideal, and further considered the application of S-universality to the relationship. The relationship in [7] determined using the fact that a generalized or inner derivation is an operator and as such, one can calculate its numerical range as well as the norm whenever applicable. Indeed, it was noted in [8] that for any operator $T \in B(H)$ and norm ideal τ in $B(H)$, $diam(W(T)) \le \| \delta_T | \tau \|$ where $'diam'$ is the diameter. Furthermore, it was shown that if $T \in B(H)$ is S-universal, and τ a norm ideal in $B(H)$, then $diam(W(T)) \le \| \delta_T | \tau \|$. In [8], Rosenblum determined the spectrum of an inner derivation, $\delta_T = TP - PT$. Kadison, Lance and in [9] the author investigated derivations δ_T acting on a general C^*-algebra and which are induced by Hermitian operators. But [10] studied a derivation δ_T acting on an irreducible C^*-algebra *B*(*H*) for all bounded linear operators on a Hilbert space H. The geometry of the spectrum of a normal operator T was used in [11] to show that the norm of a derivation is given by $\| \delta_T \| = inf\{2 \| T - \lambda \| : \lambda \in C\}$ using the geometry of the spectrum of normal operator T. However, [12] raised the question on the ability to compute the norm of a derivation on an arbitrary C^*-algebra. Research of [13] later used the density theorem to prove that the extension of derivations of a C^*- algebra to its weak-closure in $B(H)$ [14] is achieved without increasing norm. In [15] the study computed the norm of a derivation on a von Neumann algebra. Specifically, it was shown that if φ is a von Neumann algebra of operators acting on a separable Hilbert space H and $T \in \varphi$ and δ_T is the derivation induced by T, then $\| \delta_T | \varphi \| = 2\, inf\{T - Z : Z \in C\}$ where C is the center of φ [16]. Given an algebra of bounded linear endomorphisms $\mathcal{L}(\mathcal{X})$ for a real or complex vector space $\mathcal{X}$, it was shown that for each element $T \in L(\mathcal{X})$, an operator $\delta_T(A) = TA - AT$ is defined on $\mathcal{L}(\mathcal{X})$ and $\| \delta_T \| \le 2\, inf_\lambda \| T + \lambda I \|$. Furthermore if $\mathcal{X}$is a complex Hilbert space then the norm equality holds [17]. Also [18] used a method which applies to a large class of uniformly

convex spaces to show that this norm formula does not apply for ℓ^P and $L^P(0,1),\ 1 < p < \infty,\ p \neq 2$. For L^1 spaces, the formula was proved to be true in the real case but not in the complex case when the space has three or more dimensions.
The derivation constant $K(\mathcal{A})$ has been studied for unital non-commutative C^*-algebra $\mathcal{A}$ [19]. In [20] the author studied $K(M(\mathcal{A}))$ for the multiplier $M(\mathcal{A})$ for a non-unital C^*-algebra $\mathcal{A}$and obtained two results; that $K(M(\mathcal{A})) = 1$ if $\mathcal{A} = C^*(G)$ for a number of locally compact group G and $K(M(\mathcal{A})) = \frac{1}{2}$ if G is (non-abelian) amenable group. However, [21] showed that in both finite and infinite dimensional vector spaces, the norm of a generalized derivation is given by $\| \delta_{A,B} \| = \| A \| + \| B \|$ for a pair $A, B \in B(H)$. In [21] and [22], the authors showed the necessary and sufficient conditions for a derivation δ_T to be norm-attainable. Several other results exists on the inequalities of derivations and commutators on C^*-algebras. For instance [1] used a polar decomposition $T = UP$ of a complex matrix T and unitarily invariant norm $|||\cdot|||$ to prove the inequality $\||\ |UP - PU|^2\ ||| \leq \|\ |T^*T - TT^*|\ \| \leq \| UP + PU \||| UP - PU \|$. Williams [79] proved that if a commutator $TX - XA = \alpha I$ is such that A is normal, then the norm relation $\| I - (TX - XT) \| \geq \| I \|$ holds. Anderson [2], generalized Williams inequality and proved that $\| P - (TX - XT) \| \geq \| P \|$. Later, [7] proved that if T and P are normal operators, then $I - (TX - XP)\ \geq \| I \|$. The norms of derivations implemented by S-universal operators have been shown to be less than or equal to half the sum of inner derivations implemented by each operator in [7] and in particular was proved that, $\| \delta_{T,P} \| \leq \frac{1}{2}(\| \delta_{T-\lambda} \| + \| \delta_{P-\lambda} \|)$ and $\| \delta_{T-\lambda,P-\lambda} \| \leq \frac{1}{2}(\| \delta_{T-\lambda} \| + \| \delta_{P-\lambda} \|)$. Using unitaries and non-orthogonal projections, Bhatiah and Kittaneh [5] determined max-norms and numerical radii inequalities for commutators. Some authors have used the concept of classical numerical range to study different classes of matrices of operators. For instance, many alternative formulations of (p,q)-numerical range $Wp,q(A) = \{Ep((UAU^*)[Q])$ for a unitary U where $1 \leq p \leq q \leq n$ for an $n \times n$ complex matrix X, with $q \times q$ leading principle submatrix $X[q]$ and the pth elementary symmetric functions of the eigen values of $X[q]$ [8]. In [7] the author extended the results of these formulations to the generalized cases, gave alternative proofs for some of them like convexity and even derived a formula for (p,q)- numerical radius of a derivation as $r_{p,q}(T) = max\{|\mu| : \mu \in Wp,q(T)\}$. In [14] applied positive operators in the proof of a similar result. Orthogonal projections being bounded operators, have extensive uses on implementation of derivations and construction of underlying algebras of the derivations. Vasilevski [76] studied the applications of C^*-algebras constructed by orthogonal projections to Naimark's dilation theorem. In [22] the author used orthogonal projections to induce a derivation on von Neumann algebras. In [9] the researcher used mutually orthogonal projections acting on a C^*-algebra to prove that any local derivation is a derivation.

2. Basic definitions

Definition 2.2.0: An elementary operator $T \in B(H)$ is said to be norm-attainable if there exists a unit vector $x_0 \in H$, such that $\| Tx_0) \| = \| T \|$.

Definition 1.21: A Hilbert-Schmidt operator T with orthonormal basis $\{e_i : i \in I\}$ has a Hilbert-Schmidt norm $\|.\|_2$ is defined by $\| T \|_2 = (\sum_{i \in I} \| Te_i \|^2)$

Definition 2.2.1: Let Hn denote the complex vector space of all $n \times n$ Hermitian matrices, endowed with the inner product $\langle A, B \rangle = Tr(B^*A)$, where Tr(.) is the trace on the positive matrices and B^* is the adjoint of B, then:

(i). the trace norm of T, is defined by, $\| T \|_1 = \sum_{i=0}^{n} s_i T$.

(ii). the spectral norm of T, also is defined by, $\| T \| 2 = max\{s_i T\}$, where siT are the singular values of T, i.e., the eigenvalues of $| T | = (T^*T)^{\frac{1}{2}}$.

Definition 2.2.2: A tensor product of H with K is a Hilbert space P, together with a bilinear mapping $\varphi : \mathrm{H} \times \mathrm{K} \rightarrow \mathrm{P}$, such that
(i). The set of all vectors $\varphi(x, y)(x \in \mathrm{H},\ y \in \mathrm{K})$ forms a total subset of P, that is, its closed linear span is equal to P;

(ii). $\langle \varphi(x_1, y_1), \varphi(x_2, y_2) \rangle = \langle x_1, x_2 \rangle \langle y_1, y_2 \rangle$ for $x_1, x_2 \in \mathrm{H}, y_1, y_2 \in \mathrm{K}$.

We refer to the pair (P, φ) as the tensor product.

Remark 2.2.3: Let $\mathrm{X}, \mathrm{X}', \mathrm{Y}$ and Y' be vector spaces over some fields and $\mathrm{P} : \mathrm{X} \mapsto \mathrm{X}'$, and $\mathrm{Q} : \mathrm{Y} \mapsto \mathrm{Y}'$ be operators. Then there is a unique linear operator $\mathrm{P} \odot \mathrm{Q} : \mathrm{X} \otimes \mathrm{Y} \rightarrow \mathrm{X}' \otimes \mathrm{Y}'$ defined by $(\mathrm{P} \odot \mathrm{Q})(\mathrm{X} \otimes \mathrm{Y}) = \mathrm{P}(x) \otimes \mathrm{Q}(y),\ \forall x \in \mathrm{X},\ y \in \mathrm{Y}$

The function $f : \mathrm{X} \times \mathrm{Y} \rightarrow \mathrm{X}' \otimes \mathrm{Y}'$ defined by $f(x, y) = \mathrm{P}(x) \otimes \mathrm{Q}(y)$ is bilinear and so by the universal property of tensor products, there exist a unique operator $\mathrm{P} \odot \mathrm{Q}$ for which the above equation holds. The map $\mathrm{P} \odot \mathrm{Q}$ is called the tensor product of P and Q.

3. Main results

Lemma 3.0.0: Given that $P, Q, X \in B(H)$ are matricial operator on a finite dimensional separable Hilbert space H^n then $PX - XQ$ is also matricial.

Proof: Let $[p_{ij}]$, $[q_{ij}]$ and $[x_{ij}]$ denote the matrices of the operators P, Q and X respectively. Suppose that $v_i = v_1, \dots, v_n$ forms a basis of H^n over a field $\mathbb{K}$, then a simple computation shows that for $(P - Q)v_i = P\, v_i - Q\, v_i$

$$= \sum_j p_{ij} v_j - \sum_j q_{ij} v_j$$

$$= \sum_j (p_{ij} - q_{ij}) v_j$$

which can also be written more compactly as $\sum_j \gamma_{ij} v_j$ where γ_{ij} is the finite difference $p_{ij} - q_{ij}$ for every i and j. For a given $\lambda \in \mathbb{K}$ then it is also clear that $\lambda[p_{ij}] = [\lambda p_{ij}]$. We adopt the order $v_i T$ (instead of $T v_i$) for the image of an arbitrary operator T which acts on H_n for $v_i \in H_n$. Thus, $v_i TX = (v_i T)X = (\sum_j p_{ij} v_j\ X = \sum_j p_{ij} (v_j\ X)$. But $v_j X = \sum_k x_{jk} v_k$ and so by substituting in the equation above yields $vi(PX) = \sum_j p_{ij} (\sum_k x_{jk} vk) = \sum_k (p_{ij} x_{jk}) vk$ so that $[PX] = \alpha_{ij}$ where for each i and j, $\alpha_{ij} = \sum_k p_{ij} x_{jk}$. Thus, we can also find $\beta_{ij} = \sum_i x_{jk} q_{ki}$ so that $\gamma'_{ij} = \alpha_{ij} - \beta_{ij} = \sum_k p_{ij} x_{jk} - \sum_i x_{jk} q_{ki}$.

Theorem 3.0.1: Let $\delta : B(H) \to B(H)$ be a generalized derivation defined $by\ \delta_{P,Q}(X) = PX - XQ$ for orthogonal projections P and Q induced by q_n, then $\| \delta_{P,Q} \| = \{\sum | p_n |^2\}^{\frac{1}{2}} - \{\sum | q_n |^2\}^{\frac{1}{2}}$ and $\| \delta_{P,Q}(X) \| = \| P \| \| X \| - \| X \| \| Q \|$

Proof: Taking $\|f_n\| = 1$ for a fixed $P, Q \in P_0(H)$ then $\delta_{P,Q}(f_n) = pf_n - f_n q$. Suppose that p_n and q_n which induce P and Q respectively are bounded, then $\delta_{P,Q} fn = pf_n - f_n q$ can take the form of a diagonal matrix and $\sum_n (p_n f_n - f_n q_n)$ is also bounded. Now

$$\| \delta_{P,Q}(f_n) \|^2 = \| \sum_n (p_n f_n - f_n q_n) \|^2$$

$$\geq \| \sum_n p_n f_n \|^2 - \| \sum_n q_n f_n \|^2$$

$$= \sum_n | p_n |^2 \| f_n \|^2 - \sum_n | q_n |^2 \| f_n \|^2$$

$$= \{\sum_n |p_n|^2 - \sum_n | q_n |^2\}\{\sum_n \| f_n \|^2\}$$

so that on taking the supremum over both sides of the inequality gives

$$\sup\{\| Pf_n - f_n Q \| : \| f_n \| = 1\} \geq \| \delta_{P,Q}(f_n) \|$$

$$\geq \{\sum_n |p_n|^2\}^{\frac{1}{2}} - \{\sum_n |q_n|^2\}^{\frac{1}{2}}.$$

Conversely the following relation hold

$$\{\| \sum_n (p_n f_n - f_n q_n) \|\}^{\frac{1}{2}} \leq \left\{\{\sum_n |p_n|^2\}^{\frac{1}{2}} - \{\sum_n | q_n |^2\}^{\frac{1}{2}}\right\}\{\sum_n \| f_n \|^2\}^{\frac{1}{2}}$$

Which implies that the following also hold.

$$\{\| \sum_n (p_n f_n - f_n q_n) \|\} \leq \{\sum_n |p_n|^2 \| f_n \|^2 - \sum_n |q_n|^2 \| f_n \|^2\}$$

$$= \{\sum_n \| p_n f_n \|^2 - \sum_n \| q_n f_n \|^2\}$$

$$\leq \{\| \sum_n (p_n f_n - f_n q_n) \|\}$$

So

$sup\{\| Pf_n - f_n Q \| : \| f_n \| = 1\} = \| \delta_{P,Q}(f_n) \|$ and for an arbitrary $X \in B(H)$, then for $X = \sum_n X_n f_n$

$$\| \delta_{P,Q}(X) \| = \{\sum_n \| p_n \|\}^{\frac{1}{2}} \{\sum_n \| X_n f_n \|^2\}^{\frac{1}{2}} - \{\sum_n \| X_n f_n \|^2\}^{\frac{1}{2}} \{\sum_n \| q_n \|\}^{\frac{1}{2}}$$

$$= \| P \| \| X \| - \| X \| \| Q \|$$

The following is a discussion of the norms of derivations in the context of tensor product of operators. We show that indeed δ P, Q is linear and bounded in this context.

Remark 3.0.2: Suppose that $H = \ell^2$ is infinite dimensional complex Hilbert space, then ℓ^2 is unitarily invariant to the Hilbert space tensor product $\ell^2 \otimes \ell^2$. Let $P \in (H^n, H_1)$, $Q \in B(H^n, H_2)$ and an arbitrary $X : H^n \to H^n$ for $H^n = H_1 \oplus H_2 = H_{11} \oplus H_{22}$. There is a unique linear operator $P \odot X \in B(H^n \otimes H^n, H_1 \otimes H_1)$, called the tensor product of P and X satisfying $(P \odot X)(x \otimes y) = P(x) \otimes X(y)$ and similarly $(X \odot Q)(y \otimes x) = X(y) \otimes Q(x)$. Moreover, there is a unique injective linear operator $\theta : B(H^n, H_1) \otimes B(H^n, H_2) \to B(H^n \otimes H_1), B(H^n \otimes H_2)$ which satisfy $\theta(P \otimes X - X \otimes Q) = P \odot X - X \odot Q$.

Theorem 3.0.3: Let $P \in B(H^n, H_1)$, $Q \in B(H^n, H_2)$ and an arbitrary $X : H^n \to$ for $H^n = H_1 \oplus H_2 = H_{11} \oplus H_{22}$ then $\delta_{P,Q}$ is linear and bounded.

Proof: By the definition of derivations, the map $\delta_{P,Q}(X) = P \otimes X - X \otimes Q : B(H_1 \otimes H_{11}) \to (H_2 \otimes H_{22})$ is defined by

$$P \odot X(\textstyle\sum_{i=1}^n x_i \otimes y_i) - X \odot Q(\sum_{i=1}^n y_i \otimes x_i) = \sum_{i=1}^n P(x_i) \otimes X(y_i) - \sum_{i=1}^n X(x_i) \otimes Q(y_i) \text{ for all } x \in H^n.$$

Let $\alpha, \beta \in \mathbb{F}$ and $\sum_{i=1}^n x_i \otimes y_i, \sum_{i=1}^n x'_i \otimes y'_i \in H_1 \otimes H_{11}$. Then

$$P \odot X - X \odot Q\big(\alpha \textstyle\sum_{i=1}^n x_i \otimes y_i - \beta \sum_{i=1}^n x'_i \otimes y'_i\big) = (P \odot X - X \odot Q)(\alpha \sum_{i=1}^n x_i \otimes y_i) + (\beta \sum_{i=1}^n x'_i \otimes y'_i)$$

$$= (P \odot X - X \odot Q)(\alpha \textstyle\sum_{i=1}^n x_i \otimes y_i) + (P \odot X - X \odot Q)\big(\beta \sum_{i=1}^n x'_i \otimes y'_i\big)$$

$$= P \odot X(\alpha \textstyle\sum_{i=1}^n x_i \otimes y_i) - X \odot Q(\alpha \sum_{i=1}^n x_i \otimes y_i) + P \odot X\big(\beta \sum_{i=1}^n x'_i \otimes y'_i\big) - X \odot Q(\beta \displaystyle\sum_{i=1}^n x'_i \otimes y'_i$$

$$= \alpha \sum_{i=1}^n P(x_i) \otimes X(y_i) - \alpha \sum_{i=1}^n X(x_i) \otimes Q(y_i) + \beta \sum_{i=1}^n P(x'_i) \otimes X(y'_i) - \beta \sum_{i=1}^n X(x'_i) \otimes Q(y'_i)\, \alpha P \odot X\left(\sum_{i=1}^n x_i \otimes y_i\right) - \alpha X \odot Q\left(\sum_{i=1}^n x_i \otimes y_i\right) + \beta P \odot X\left(\sum_{i=1}^n x'_i \otimes y'_i\right) - \beta X \odot Q\left(\sum_{i=1}^n x'_i \otimes y'_i\right)$$

$$= \alpha(P \odot X - X \odot Q)(\sum_{i=1}^n x_i \otimes y_i) + \beta(P \odot X - X \odot Q)(n \sum_{i=1}^n x'_i \otimes y'_i)$$

Now for the case of boundedness,

$$\| (P \odot X - X \odot Q)(\alpha \textstyle\sum_{i=1}^n x_i \otimes y_i) \| = |\sum_{i=1}^n P(x_i) \otimes X(y_i) - \sum_{i=1}^n X(x_i) \otimes Q(y_i)|$$

$$\leq \| \textstyle\sum_{i=1}^n P(x_i) \otimes X(y_i) - \sum_{i=1}^n X(x_i) \otimes Q(y_i) \|$$

$$\leq \| \textstyle\sum_{i=1}^n P(x_i) \otimes X(y_i) \| + \| \sum_{i=1}^n X(x_i) \otimes Q(y_i) \|$$

$$\leq \| \textstyle\sum_{i=1}^n P(x_i) \| \| X(y_i) \| + \| \sum_{i=1}^n X(x_i) \| \| X(y_i) \|$$

$$\leq \| \textstyle\sum_{i=1}^n P \| \| (x_i) \| \| X \| \| (y_i) + \| \sum_{i=1}^n X \| \| (x_i) \| \| Q \| \|$$

$$\leq \| P \| \| X \| \| \textstyle\sum_{i=1}^n \| x_i y_i \| + \| X \| \| Q \| \sum_{i=1}^n \| x_i y_i \|$$

$$= (\| P \| \| X \| + \| X \| \| Q \|) \textstyle\sum_{i=1}^n \| x_i \| \| y_i \|.$$

Letting $(\| P \| \| X \| + \| X \| \| Q \|) = M$, thus M is the upper bound for $PX - XQ$

Theorem 3.0.4: Let $X \in B(H)$ and orthogonal projections $P, Q \in B(H)$ then $\| P \odot X - X \odot Q \| = \| P \| \| X \| - \| X \| \| Q \|$.

Proof:

$$\| P \odot X - X \odot Q \| = sup \left\{ \| \sum_{i=1}^n (x_i \otimes y_i) \| = 1 \,\| P \odot X - X \odot Q(\sum_{i=1}^n x_i \otimes y_i) \| \right\}$$

$$\leq \sup \left\{ \| \sum_{i=1}^n (x_i \otimes y_i) \| = 1 \,\| P \| \| X \| \| (\sum_{i=1}^n x_i \otimes y_i) \| - \| X \| \| Q \| \; \| (\sum_{i=1}^n x_i \otimes y_i)) \| \| \sum_{i=1}^n x_i \otimes y_i \| \right\}$$

$$= \| P \| \| X \| - \| X \| \| Q \|$$

Conversely,

$$\| P \odot X - X \odot Q \| = sup\{\| P \odot X(\sum_{i=1}^{n} x_i \otimes y_i) - X \odot Q(\sum_{i=1}^{n} x_i \otimes y_i) \| \ \forall \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y\}$$ and

$\sum_{i=1}^{n} x_i \otimes y_i) \neq 0.$

Then

$$\| P \odot X - X \odot Q \| \geq \left\{ \frac{\| P \odot X(\sum_{i=1}^{n} x_i \otimes y_i) - X \odot Q(\sum_{i=1}^{n} x_i \otimes y_i) \|}{\|(\sum_{i=1}^{n} x_i \otimes y_i)\|} \ \forall \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y \right\}$$

And $\sum_{i=1}^{n} x_i \otimes y_i) \neq 0 = \| P \| \| X \| - \| X \| \| Q \|.$

Thus $|P \odot X - X \odot Q \| \geq \| P \| \| X \| - \| X \| \| Q \|.$

In the sequel, we will consider inequalities for the norms of derivation discussed. The inequalities considered will be on generalized derivations and the results generalize to the cases of inner derivations.

Theorem 3.0.5: Suppose that $P, Q \in P_0(H)$ are matricial operators, then

$$\| \delta_{P,Q}(X) \|^2 = (\sum_{ij=1}^{2} \sum_{ij=1}^{2} |p_i x_{ij} - x_{ij} q_j|^2)^{\frac{1}{2}}_{(i=j)} + (\sum_{ij=1}^{2} \sum_{ij=1}^{2} |p_i x_{ij} - x_{ij} q_j|^2)^{\frac{1}{2}}_{(i=j, i\leftrightarrow j)} + (\sum_{j=1}^{2} |q_j x_{3j}|^2)^{\frac{1}{2}} \text{ on } \oplus H_3$$

Proof: Suppose that P and Q are positive diagonal $n \times n$ matrices with eigenbases p_n and qn respectively for $n \geq 1$, with $p_n(1 - p_n^*) = 0$ and $q_n(1 - q_n^*) = 0$. Given arbitrary $X \in B(H)$, then,

$$P = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, Q = \begin{bmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ and an arbitrary $$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$, with $p_1 \geq p_2 \geq 0$ then

$$PX - XP = \begin{bmatrix} p_1 x_{11} - x_{11} p_1 & p_1 x_{12} - x_{12} p_2 & p_1 x_{13} \\ p_2 x_{21} - x_{21} p_1 & p_2 x_{22} - x_{22} p_2 & p_2 x_{23} \\ -x_{31} p_1 & -x_{32} p_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} p_1 x_{11} - x_{11} p_1 & 0 & 0 \\ 0 & p_2 x_{22} - x_{22} p_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & p_1 x_{12} - x_{12} p_2 & p_1 x_{13} \\ p_2 x_{21} - x_{21} p_1 & 0 & p_2 x_{23} \\ -x_{31} p_1 & -x_{32} p_2 & 0 \end{bmatrix}$$

So that for a commutative $B(H)$ then

$$PX - XP = \begin{bmatrix} 0 & p_1 x_{12} - x_{12} p_2 & p_1 x_{13} \\ p_2 x_{21} - x_{21} p_1 & 0 & p_2 x_{23} \\ -x_{31} p_1 & -x_{32} p_2 & 0 \end{bmatrix}.$$

Now

$$PX + XP = \begin{bmatrix} p_1 x_{11} + x_{11} p_1 & p_1 x_{12} + x_{12} p_2 & p_1 x_{13} \\ p_2 x_{21} + x_{21} p_1 & p_2 x_{22} + x_{22} p_2 & p_2 x_{23} \\ x_{31} p_1 & x_{32} p_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} p_1 x_{11} + x_{11} p_1 & 0 & 0 \\ 0 & p_2 x_{22} + x_{22} p_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & p_1 x_{12} + x_{12} p_2 & p_1 x_{13} \\ p_2 x_{21} + x_{21} p_1 & 0 & p_2 x_{23} \\ x_{31} p_1 & x_{32} p_2 & 0 \end{bmatrix}$$

and

$$QX - XQ = \begin{bmatrix} q_1 x_{11} - x_{11} q_1 & q_1 x_{12} - x_{12} q_2 & q_1 x_{13} \\ q_2 x_{21} - x_{21} q_1 & q_2 x_{22} - x_{22} q_2 & q_2 x_{23} \\ -x_{31} q_1 & -x_{32} q_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} q_1 x_{11} - x_{11} q_1 & 0 & 0 \\ 0 & q_2 x_{22} - x_{22} q_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & q_1 x_{12} - x_{12} q_2 & q_1 x_{13} \\ q_2 x_{21} - x_{21} q_1 & 0 & q_2 x_{23} \\ -x_{31} q_1 & -x_{32} q_2 & 0 \end{bmatrix}$$

Similarly for a commutative $B(H)$ then $$QX - XQ = \begin{bmatrix} 0 & q_1 x_{12} - x_{12} q_2 & q_1 x_{13} \\ q_2 x_{21} - x_{21} q_1 & 0 & q_2 x_{23} \\ -x_{31} q_1 & -x_{32} q_2 & 0 \end{bmatrix}.$$

Now

$$QX + XQ = \begin{bmatrix} q_1x_{11} + x_{11}q_1 & q_1x_{12} + x_{12}q_2 & q_1x_{13} \\ q_2x_{21} + x_{21}q_1 & q_2x_{22} + x_{22}q_2 & q_2x_{23} \\ x_{31}q_1 & x_{32}q_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} q_1x_{11} + x_{11}q_1 & 0 & 0 \\ 0 & q_2x_{22} + x_{22}q_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & q_1x_{12} + x_{12}q_2 & q_1x_{13} \\ q_2x_{21} + x_{21}q_1 & 0 & q_2x_{23} \\ x_{31}q_1 & x_{32}q_2 & 0 \end{bmatrix}$$

We obtain an operator

$$(PX - XQ) = \begin{bmatrix} p_1x_{11} - x_{11}q_1 & p_1x_{12} - x_{12}q_2 & p_1x_{13} \\ p_2x_{21} - x_{21}q_1 & p_2x_{22} - x_{22}q_2 & 0 \\ -q_1x_{31} & -q_2x_{32} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} p_1x_{11} - x_{11}q_1 & 0 & 0 \\ 0 & p_2x_{22} - x_{22}q_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & p_1x_{12} - x_{12}q_2 & p_1x_{13} \\ p_2x_{21} - x_{21}q_1 & 0 & 0 \\ -q_1x_{31} & -q_2x_{32} & 0 \end{bmatrix}$$

Now on introducing the norm function to the equality results into the norm inequality;

$$\left\|\begin{bmatrix} p_1x_{11} - x_{11}q_1 & p_1x_{12} - x_{12}q_2 & p_1x_{13} \\ p_2x_{21} - x_{21}q_1 & p_2x_{22} - x_{22}q_2 & 0 \\ -q_1x_{31} & -q_2x_{32} & 0 \end{bmatrix}\right\| \le \left\|\begin{bmatrix} p_1x_{11} - x_{11}q_1 & 0 & 0 \\ 0 & p_2x_{22} - x_{22}q_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\|$$

$$+ \left\|\begin{bmatrix} 0 & p_1x_{12} - x_{12}q_2 & 0 \\ p_2x_{21} - x_{21}q_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\| - \left\|\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -q_1x_{31} & -q_2x_{32} & 0 \end{bmatrix}\right\|$$

Application of Hilbert-Schmidt norm to this, gives us the following

$$(\Sigma_{ij=1}^{2}\Sigma_{ij=1}^{2}|p_ix_{ij} - x_{ij}q_j|^2)_{(i=j)}^{\frac{1}{2}} + (\Sigma_{ij=1}^{2}\Sigma_{ij=1}^{2}|p_ix_{ij} - x_{ij}q_j|^2)_{(i\ne j,i\leftrightarrow j)}^{\frac{1}{2}} + (\Sigma_{j=1}^{2}|q_jx_{3j}|^2)^{\frac{1}{2}}$$

Lemma 3.0.6: Let $P \in P_0(H)$ and X is compact, then $s_j(PX) = s_j(XP) \le \| X \| s_j(P)$

Proof: $s_j(PX) = s_j(XP)$ is immediate from the commutativity of the singular values and $s_j(XP) \le \| X \| s_j(P)$ follows from the correspondence $s_j(.) \equiv \|.\|$, and the inequality, $||| PX ||| \le \| P \| \| X \|$.

Theorem 3.0.7: Let $B(H)$ be a C^*-algebra, $P_0(H^n)$ a commutative subalgebra of $B(H)$ and a map $\delta_{P,Q}$, such that $\delta_{P,Q} : P_0(H^n) \to B(H)$. Let $\delta_{P,Q} : M_n(P_0(H^n)) \to M_n(H^n)$ be a linear map between matricial operator spaces $M_n(P_0(H^n))$ and $M_n(H^n)$. For n-tuples of $\delta_{P,Q}$, whereby $\delta_n : M_n[P_0(H^n)] \to M_n[B(H)]$, then $\delta_n[(P,Q)] = [\delta(P,Q)], \forall P,Q \in M_n[P_0(H^n)]$ and $[P] = [P_1,P_2]$, $[Q] = [Q_1,Q_2]$. Moreover, $\| \delta_{P,Q} \| \le \| \delta_{P,Q} \|_{CB}$ holds.

Proof: We apply diagonal matrices $[P]$ and $[Q]$. For $n = 1$, then by definition of δ_n, δ_1 and δ are coincidental [20] hence, $\| \delta \| = \| \delta 1 \|$. We now proceed to give proofs when $n = 2$ and when $n = 3$. For $n = 2$, let $[P], [Q] \in M_2[P_0(H^n)], j,k = 1,2,$ then for $\delta_2 : M_2[P_0(H^n)] \to M_2[B(H)]$, we now have,

$$\delta_2 P,Q = \delta_2\left(\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}\begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} P_1X - XQ_1 & 0 \\ 0 & P_2X - XQ_2 \end{bmatrix}$$

$$= \begin{bmatrix} \delta_{(P_1,Q_1)} & 0 \\ 0 & \delta_{(P_2,Q_2)} \end{bmatrix}$$ and

$$\left\|\left(\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}\begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}\right)\right\| = \left\|\begin{bmatrix} P_1X - XQ_1 & 0 \\ 0 & P_2X - XQ_2 \end{bmatrix}\right\|$$

$$= \left\|\begin{bmatrix} \delta_{(P_1,Q_1)} & 0 \\ 0 & \delta_{(P_2,Q_2)} \end{bmatrix}\right\|$$

$$= [\ \Sigma_{j=1}^{2}\Sigma_{k=1}^{2} \ \| \delta((P_j,Q_k) \|^2]^{\frac{1}{2}}$$ (Hilbert-Schmidt norm)

$$= (\| \delta((P_1, Q_1) \|^2 + \| \delta((P_2, Q_2) \|^2)^{\frac{1}{2}}$$

$$\geq [\| \delta((P_1, Q_1) \|^2]^{\frac{1}{2}}$$

$$= \| \delta((P_1, Q_1) \|$$

$$= \| \delta_1((P_1, Q_1) \|.$$

Therefore,

$$\| \delta_2 \| = sup\{\|\delta_2(PQ) : [PQ] \in M_2[P_0(H^n)\|\}$$

$$\geq sup\{\| \delta_1((P_1, Q_1) \|\} = \| \delta_1 \|$$

and hence $\| \delta_2 \| = \| \delta_1 \|$.

When $n = 3, \ \delta 3 \begin{bmatrix} P_1X - XQ_1 & 0 & 0 \\ 0 & P_2X - XQ_2 & 0 \\ 0 & 0 & P_3X - XQ_3 \end{bmatrix}$

$$= \begin{bmatrix} \delta_{(P_1,Q_1)} & 0 & 0 \\ 0 & \delta_{(P_2,Q_2)} & 0 \\ 0 & 0 & \delta_{(P_3,Q_3)} \end{bmatrix}$$

which implies that

$$\left\| \delta_3 \begin{bmatrix} P_1X - XQ_1 & 0 & 0 \\ 0 & P_2X - XQ_2 & 0 \\ 0 & 0 & P_3X - XQ_3 \end{bmatrix} \right\| = \left\| \begin{bmatrix} \delta_{(P_1,Q_1)} & 0 & 0 \\ 0 & \delta_{(P_2,Q_2)} & 0 \\ 0 & 0 & \delta_{(P_3,Q_3)} \end{bmatrix} \right\|$$

$$= [\sum_{j=1}^{3} \sum_{k=1}^{3} \| \delta((P_j, Q_k) \|^2]^{\frac{1}{2}}$$

$$= (\| \delta((P_1, Q_1) \|^2 + \| \delta((P_2, Q_2) \|^2 + \| \delta((P_3, Q_3) \|^2)^{\frac{1}{2}}$$

$$= [\sum_{j=1}^{3} \sum_{k=1}^{3} \| \delta((P_j, Q_k) \|^2]^{\frac{1}{2}}$$

$$= \| \delta([\delta((P_j, Q_k)]) \|.$$

This implies that

$$\| \delta_3 \| = sup\{\| \delta_3[\delta((P_j, Q_k)] : [\delta((P_j, Q_k)] \to M_3[P_0(H^n)] \|\} \geq sup\{\| \delta_2[\delta((P_j, Q_k)] : [\delta((P_j, Q_k)] \to M_2[P_0(H^n)]||\} = \| \delta_2 \|$$

and therefore, $\| \delta_3 \| \geq \| \delta_2 \|$.

Lastly, consider $\delta_{n+1} : M_{n+1}[P_0(H^n)] \to M_{n+1}[B(H)]$ defined by $\delta_{n+1}[\delta((P_j, Q_k)] = [\delta((P_j, Q_k)]$ for all $j, k = 1, \ldots, n+1$. We obtain,

$$\| \delta_{n+1}[(PQ)_{j,k}] \| = \| [\delta(PQ)_{j,k}] \|$$

$$= [\sum_{j=1}^{n+1} \sum_{k=1}^{n+1} \| \delta((P_j, Q_k) \|^2]^{\frac{1}{2}}$$

$$\geq [\sum_{j=1}^{n} \sum_{k=1}^{n} \| \delta((P_j, Q_k) \|^2]^{\frac{1}{2}}$$

$$= \| \delta_n[\delta((P_j, Q_k)] \|.$$

Therefore, on taking supremum on both sides of the inequality above we get $\| \delta_{n+1} \| = \| \delta_n \|$. Application of the property of complete boundedness of the norm of δ, we further get $\| \delta \|_{CB} = sup\{\| \delta_n \| \ : n \in \mathbb{N}\}$which implies that $\| \delta \|_{CB} = \| \delta_n \| \ \forall \ n \in \mathbb{N}$. Therefore, $\| \delta \| = \| \delta \|_{CB}$, this completes the claim.

Example 3.0.8: Let $\delta : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be a derivation defined by $\delta_{P,Q}(X) = PX - XQ$. Let an operator P, be defined by $P(e_j) = e_j$ on a finite dimensional Hilbert space H, for an orthonormal basis $e_j, j = 1,2\ldots$

We can then set the matrix for an arbitrary operator X and that of P as,

$$X = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \; P = \begin{bmatrix} e_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

It is clear by simple calculation that

$$PX - XP = \begin{bmatrix} e_1 x_{11} - x_{11} e1\, 0 & e_1 x_{12} \\ -x_{21} p_1 & 0 \end{bmatrix}.$$

Now suppose that H has a unique direct decomposition given by $H = ranP \oplus kerP$ and e_1 is an identity in the range of P,

then $PX - XP$ becomes $PX - XP = \begin{bmatrix} 0 & e_1 x_{12} \\ -x_{21} e_1 & 0 \end{bmatrix}$. We can find a unitary $U = \begin{bmatrix} e_1 & 0 \\ 0 & -e_2 \end{bmatrix}$ such that

$$\begin{bmatrix} 0 & e_1 x_{12} \\ -x_{21} e_1 & 0 \end{bmatrix} = \frac{1}{2}(UX - XU)$$

$$= \frac{1}{2}(UX - XU^*)$$

By triangle inequality,

$$\frac{1}{2} \| (UX - XU) \| \leq \frac{1}{2} \| (UX + XU) \| \leq \frac{1}{2} \| UX \| + \frac{1}{2} \| XU \| = \| XU \| = \| X \|.$$

Now considering another operator Q similar to P, we can get another orthonormal basis $f_j, j = 1,2\ldots$ such that Q is defined by

$$Q = \begin{bmatrix} f_j \cdot & 0 \\ 0 & 0 \end{bmatrix}.$$

Let also $\| X \| = \{\sum_n | \alpha_n |^2\}^{\frac{1}{2}} = 1, \; \| PX \| = \{\sum_j | e_j |^2\}^{\frac{1}{2}} = P \, \| QX \| = \{\sum_j | f_j |^2\}^{\frac{1}{2}} = Q$ and so

$$\| PX - XQ \| \leq \| PX + XQ \| \leq \| PX \| + \| XQ \|.$$

Lemma 3.0.9: Suppose that for an arbitrary $X \in B(H)$ and $P_1X, \; P_2X, \; XQ_1, \; XQ_2 \in C_2$ then,

$n^{p-1} \sum_{i=1}^{2} \| P_i \|_p^p \leq \| \sum_{i=1}^{2} P_i X_i \|_p^p \leq \| \; P_i X_i \|_p^p$ for $0 < p \leq \infty$ and the reverse inequalities hold for $1 \leq p < \infty$.

Proof: If a_1 and a_2, are nonnegative real eigenvalues for P_1 and P_2, then

$n^{p-1} \sum_{i=1}^{n} a_i^p \leq (\sum_{i=1}^{n} a_i)^p \leq \sum_{i=1}^{n} a_i^p$. The inequalities follow, respectively, from the concavity of the function

$f(t) = t^p, \; t \in [0,\infty)$ for $0 < p \leq 1$, and the convexity of the function $f(t) = t^p, \; t \in [0,\infty)$ for $1 \leq p < \infty$.

Proposition 3.1.0: Let $P = P_1, \; P_2, \; Q = Q_1, \; Q_2 \in C_p$ and an arbitrary $X = X_1, \; X_2 \in B(H)$ for some $p > 0$. Then

$$\sum_{i,j=1}^{2} \| \; P_i X_i - P_j X_i \|_p^p + \sum_{i,j=1}^{2} \| \; X_i Q_i - X_j Q_j \|_p^p \, p + \sum_{i,j=1}^{2} \| X_i - X_j \|_p^p \geq (D_{PX-XQ}^{\frac{p-2}{2}} \sum_{i,j=1}^{2} \| \; P_i X_i - X_j Q_j \|_p^p + D_{Q-X}^{\frac{p-2}{2}} \sum_{i,j=1}^{2} \| X_i Q_i - X_j \|_p^p + D_{X-P}^{\frac{p-2}{2}} \sum_{i,j=1}^{2} \| X_i - P_j X_j \|_p^p) - (\| \sum_{i,j=1}^{2} (P_i X_i - X_j Q_i) \|_p^p + \| \sum_{i=1}^{2} (X_i Q_i - X_i) \|_p^p + \| \sum_{i=1}^{2} (X_i - P_i X_i) \|_p^p)$$

for $0 < p < 2$.

Proof: We define a constant D_P by $D_P = \sum_{i=1}^{n} \pi(P_i X_i)$

Where,

$$\sum_{i=1}^{2} \pi(P_i X_i) = \begin{cases} 1, & (P_i X_i) \neq 0; \\ 0, & (P_i X_i) = 0, \end{cases}$$

and $\quad D_P = \sum_{i=1}^{2} \pi(X_i Q_i)$

where

$$\pi(X_i Q_i) = \begin{cases} 1, & (X_i Q_i) \neq 0; \\ 0, & (X_i Q_i) = 0 \end{cases}$$

We prove the case for 0 < p < 2 and infer the result onto the other cases. We have

$$\sum_{i,j=1}^{2} \| P_iX_i - X_jQ_j \|_p^p + \sum_{i,j=1}^{2} \| X_iQ_i - X_jQ_j \|_p^p + \sum_{i,j=1}^{2} \| X_i - X_j \|_p^p + \Big(\| \sum_{i=1}^{2}(P_iX_i - X_iQ_i) \|_p^p + \| \sum_{i=1}^{2}(X_iQ_i - X_i) \|_p^p + \| \sum_{i=1}^{2}(X_i - P_iX_i) \|_p^p\Big) = 2\Big(\sum_{1<i<j<2} \| P_iX_i - P_jX_j \|_p^p + \sum_{1<i<j<2} \| X_iQ_i - X_jQ_j \|_p^p + \sum_{1<i<j<2} \| X_i - X_j \|_p^p\Big) + \Big(\| \sum_{i=1}^{2}(P_iX_i - X_iQ_i) \|_p^p + \| \sum_{i=1}^{2}(X_iQ_i - X_i) \|_p^p + \| \sum_{i=1}^{2}(X_i - P_iX_i) \|_p^p\Big) = 2\Big(\sum_{1<i<j<2} \| |P_iX_i - P_jX_j|^2 \|_{p/2}^{p/2} + \sum_{1<i<j<2} \| |X_iQ_i - X_jQ_j|^2 \|_{p/2}^{p/2} + \sum_{1<i<j<2} \| |X_i - X_j|^2 \|_{p/2}^{p/2}\Big) + \Big(\| \sum_{i=1}^{2}|P_iX_i - X_iQ_i|^2 \|_{p/2}^{p/2} + \| |\sum_{i=1}^{2}(X_iQ_i - X_i)|^2 \|_{p/2}^{p/2} + \| \sum_{i=1}^{2}|X_i - P_iX_i|^2 \|_{p/2}^{p/2}\Big) \geq \Big\|\sum_{1<i<j<2}|P_iX_i - P_jX_j|^2 + \sum_{1<i<j<2}|X_iQ_i - X_jQ_j|^2 + \sum_{i=1}^{2}|(P_iX_i - X_iQ_i)|^2\Big\|_{p/2}^{p/2} + \Big\|\sum_{1<i<j<2}|X_iQ_i - X_jQ_j|^2 + \sum_{1<i<j<2}|X_i - X_j|^2 + \sum_{i=1}^{2}|(X_iQ_i - X_i)|^2\Big\|_{p/2}^{p/2} + \Big\|\sum_{1<i<j<2}|X_i - X_j|^2 + \sum_{1<i<j<2}|P_iX_i - P_jX_j|^2 + \sum_{i=1}^{2}|X_i - P_iX_i|^2\Big\|_{p/2}^{p/2} = \Big\| \sum_{i,j=1}^{2}|P_iX_i - X_jQ_j|^2\Big\|_{p/2}^{p/2} + \Big\|\sum_{i,j=1}^{2}|X_iQ_i - X_j|^2\Big\|_{p/2}^{p/2} + \Big\|\sum_{i,j=1}^{2}|X_i - P_jX_j|^2\Big\|_{p/2}^{p/2} \geq D_{PX-XQ}^{\frac{p}{2}-1} \sum_{i,j=1}^{2}\big\| P_iX_i - X_jQ_j|^2\big\|_{p/2}^{p/2} + D_{XQ-X}^{\frac{p}{2}-1} \sum_{i,j=1}^{2}\big\|| X_iQ_i - X_j|^2\big\|_{p/2}^{p/2} D_{X-PX}^{\frac{p}{2}-1}\sum_{i,j=1}^{2}\big\||Xi - P_jX_j|^2\big\|_{p/2}^{p/2} = D_{PX-XQ}^{\frac{p}{2}-1} \sum_{i,j=1}^{2}\big\| P_i - Q_j\big\|_p^p + D_{X-QX}^{\frac{p}{2}-1}\sum_{i,j=1}^{2}\big\| X_iQ_i - X_j\big\|_p^p + D_{X-PX}^{\frac{p}{2}-1}\sum_{i,j=1}^{2}\big\| X_i - P_jX_j\big\|_p^p$$

Proposition 3.1.1: Let $P_1, P_2, Q_1, Q_2 \in C_P$ for some $p > 0$. Then

$$\sum_{i,j=1}^{2}\big\|P_iX_i - P_jX_j\big\|_p^p + \sum_{i,j=1}^{2}\big\|X_iQ_i - X_jQ_j\big\|_p^p \geq 2.2^{p-2}\sum_{i,j=1}^{2}\big\|P_iX_i - X_jQ_j\big\|_p^p - 2\Big\|\sum_{i=1}^{2}(P_iX_i - X_iQ_i)\Big\|^p \text{ for } 0 < p \leq 2.$$

Proof: We set $D_{Q-X}^{\frac{p-2}{2}}\sum_{i,j=1}^{2} = 2\, D_{Q}^{\frac{p-2}{2}} \sum_{i=1}^{2}\|X_iQ_i\|_p^p,\ D_{X-P}^{\frac{p-2}{2}}\sum_{i,j=1}^{2}\big\|X_i - P_jX_j\big\|_p^p = 2D_{P}^{\frac{p-2}{2}} \sum_{i=1}^{2}\|P_iX_i\|_p^p.$

Now

$$0 \leq \sum_{i,j=1}^{2}\big\|P_iX_i - P_jX_j\big\|_p^p + \sum_{i,j=1}^{2}\big\|X_iQ_i - X_jQ_j\big\|_p^p - D_{PX-XQ}^{\frac{p-2}{2}} \sum_{i,j=1}^{2}\big\|P_iX_i - X_jQ_j\big\|_p^p - 2\Big(D_{XQ}^{\frac{p-2}{2}} \sum_{i=1}^{2}\| X_iQ_i\|_p^p + D_{PX}^{\frac{p-2}{2}} \sum_{j=1}^{2} \| P_j X_j \|_p^p\Big) + \Big(\sum_{i=1}^{2}(P_iX_i - X_iQ_i)_p^p + \Big\|\sum_{i=1}^{2} X_iQ_i\Big\|_p^p + \Big\|\sum_{i=1}^{2}P_iX_i\Big\|_p^p\Big) = \sum_{i,j=1}^{2}\big\|P_iX_i - P_jX_j\big\|_p^p + \sum_{i,j=1}^{2}\big\|X_iQ_i - X_jQ_j\big\|_p^p - 2D_{PX-XQ}^{\frac{p-2}{2}} \sum_{i,j=1}^{2}\big\|P_iX_i - X_jQ_j\big\|_p^p + 2\Big\|\sum_{i=1}^{2}(P_iX_i - X_iQ_i)\Big\|_p^p + \Big(D_{PX-XQ}^{\frac{p-2}{2}}\sum_{i,j=1}^{2}\big\|P_iX_i - X_jQ_j\big\|_p^p - \sum_{i=1}^{2}(P_iX_i - X_iQ_i)_p^p\Big) + \Big(\Big\|\sum_{i=1}^{2}X_iP_i\Big\|_p^p - 2D_{PX}^{\frac{p-2}{2}} \sum_{i=1}^{2}\|P_iX_i\|_p^p\Big) + \Big(\sum_{i=1}^{2}\|X_iQ_i\|_p^p - 2D_{XQ}^{\frac{p-2}{2}} \sum_{i=1}^{2}\|X_iQ_i\|_p^p\Big). D_{P-Q}^{\frac{p-2}{2}} \sum_{i,j=1}^{2}\big\|P_iX_i - X_jQ_j\big\|_p^p - \Big\|\sum_{i=1}^{2}(P_iX_i - X_iQ_i)\Big\|_p^p = D_{PX-XQ}^{\frac{p}{2}-1}\sum_{i,j=1}^{2}\big\||P_iX_i - X_jQ_j|^2\big\|_{p/2}^{p/2} - \Big\|\sum_{i=1}^{2}|P_iX_i - X_iQ_i|^2\Big\|_{p/2}^{p/2} \leq 0.$$

Since $2D_{P}^{\frac{p}{2}-1}$ is greater than or equal to 1, we deduce from lemma 4.20 that

$$\Big\|\sum_{i=1}^{2} P_iX_i\Big\|_p^p - 2D_{PX}^{\frac{p}{2}-1} \sum_{i=1}^{2}\|P_iX_i\|_p^p \leq \Big\|\sum_{i=1}^{2}P_iX_i\Big\|_p^p - \sum_{i=1}^{2}\|X_iP_i\|_p^p \leq 0.$$

Similarly, we have $\Big\|\sum_{i=1}^{2} X_iQ_i\Big\|_p^p \leq 2D_{PX}^{\frac{p-2}{2}} \sum_{i=1}^{2}\|X_iQ_i\|_p^p$. It therefore implies that

$$\sum_{i,j=1}^{2}\big\|P_iX_i - P_jX_j\big\|_p^p + \sum_{i,j=1}^{2}\big\|X_iQ_i - X_jQ_j\big\|_p^p \geq 2D_{PX-XQ}^{\frac{p-2}{2}} \sum_{i,j=1}^{2}\big\|X_iQ_i - X_jQ_j\big\|_p^p - 2\Big\|\sum_{i=1}^{2}(P_iX_i - X_iQ_i)\Big\|_p^p \geq 2.2^{p-2}\sum_{i,j=1}^{2}\big\|P_iX_i - X_jQ_j\big\|_p^p - 2\Big\|\sum_{i=1}^{2}(P_iX_i - X_iQ_i)\Big\|_p^p \geq D_{PX-XQ}.$$

4. Conclusion

In this paper, we have shown that the norm of a derivation, induced by orthogonal projections via tensor product is linear, bounded and continuous. Furthermore, we have inequalities of such a derivation induced by n-tupled orthogonal projections.

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