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On Homogeneous Bi-quadratic Diophantine Equation with Five Unknowns $2(x-y)(x^3+y^3)=4^{2n}(z^2-w^2)T^2$

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Abstract

The Bi-quadratic Diophantine equation with five unknowns given by $2(x-y)(x^3+y^3)=4^{2n}(z^2-w^2)T^2$ is analyzed for its non-zero distinct integer solutions. A few interesting relations among the solutions are presented.

Keywords: Bi-Quadratic Equation with Five Unknowns, Homogeneous Bi-Quadratic, Integral Solutions

1. Introduction

The Diophantine equations are rich in variety and offer an unlimited field for research [1-3]. In particular refer [4-14] for a few problems on Biquadratic equation with 2, 3, 4 and 5 unknowns. This paper concerns with yet another interesting Biquadratic Diophantine equation with five variables given by $2(x-y)(x^3+y^3)=4^{2n}(z^2-w^2)T^2$ for determining its infinitely many non-zero distinct integral solutions. A few interesting relations among the solutions are presented.

2. Method of analysis

The homogeneous Bi-quadratic diophantine equation with five variables under consideration is

$$2(x-y)(x^3+y^3)=4^{2n}(z^2-w^2)T^2 \tag{1}$$

Method 1

Introducing the linear transformations

$$x=4^n(u+v), y=4^n(u-v), T=4^nP, z=2u+v, w=2u-v, u\neq v \tag{2}$$

in (1), it reduces to the equation

$$u^2+3v^2=P^2 \tag{3}$$

whose solutions may be taken as

$$v=2ab, u=3a^2-b^2, P=3a^2+b^2 \tag{4}$$

In view of (2), the corresponding integer solutions to (1) are given by

$$\left.\begin{array}{l} x = x(a,b) = 4^n(3a^2 - b^2 + 2ab) \\ y = y(a,b) = 4^n(3a^2 - b^2 - 2ab) \\ z = z(a,b) = 2(3a^2 - b^2 + ab) \\ w = w(a,b) = 2(3a^2 - b^2 - ab) \\ T = T(a,b) = 4^n(3a^2 + b^2) \end{array}\right) \quad (5)$$

Relations among the solutions:

1. $4x + 4^n w = 3 * 4^n * z$

2. Each of the following expressions is a perfect square:

$T^2(a,b) - x(a,b)y(a,b)),$

$T^2(a,b) + 3x(a,b)y(a,b))$

3. Each of the following expressions is a nasty number:

$78(T^2(a,b) + 3 * 4^{2n} z(a,b)w(a,b)),$

$78(4T^2(a,b) - 4^{2n} z(a,b)w(a,b))$

Note 1:
Apart from (2), one may consider the following transformations

$x = 4^n(u+v), y = 4^n(u-v), T = 4^n P, z = u + 2v, w = u - 2v, u \neq v$

$x = 4^n(u+v), y = 4^n(u-v), T = 4^n P, z = 2uv + 1, w = 2uv - 1, u \neq v$

leading to two different solutions to (1).

Note 2:
Write (3) as a system of double equations as in Table:1 below:

Table 1: System of double equations

System	I	II
$P + u$	$3v^2$	v^2
$P - u$	1	3

Solve each of the above two systems for P, u, v. Then, from (2), one obtains the corresponding integer solutions to (1). For simplicity, the integer solutions to (1) are exhibited

Below:
Solutions from system I:

$x = 4^n * (6k^2 + 8k + 1),$

$y = 4^n * (6k^2 + 4k - 1),$

$T = 4^n * (6k^2 + 6k + 1),$

$z = (12k^2 + 14k + 1),$

$w = (12k^2 + 10k - 1)$

Solutions from system II:

$x = 4^n * (2k^2 + 4k),$

$y = 4^n * (2k^2 - 2),$

$T = 4^n * (2k^2 + 2k + 2),$

$z = (4k^2 + 6k - 1),$

$w = (4k^2 + 2k - 3)$

Note 3:
Rewrite (3) as

$$u^2 + 3v^2 = P^2 = P^2 * 1 \quad (6)$$

Assume

$$P = a^2 + 3b^2 \quad (7)$$

Write 1 on the R.H.S. of (6) as

$$1 = \frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4} \tag{8}$$

Substituting (7) & (8) in (6) and employing the method of factorization, consider

$$u + i\sqrt{3}v = \frac{(a+i\sqrt{3}b)^2(1+i\sqrt{3})}{2} \tag{9}$$

Equating the real &imaginary parts in (9), the values of u and v are obtained.
Since our interest is on finding integer solutions, replace a by 2A, b by 2B
in the above resulting values of u, v and (7). In view of (2), the corresponding integer solutions to (1) are as follows:

$$\begin{aligned} x &= 4^{n+1} * (A^2 - 3B^2 - 2AB), \\ y &= -4^{n+2} * (AB), \\ T &= 4^{n+1} * (A^2 + 3B^2), \\ z &= 2(3A^2 - 9B^2 - 10AB), \\ w &= 2(A^2 - 3B^2 - 14AB) \end{aligned}$$

Remark:
In addition to (8), 1 may expressed as below:

$$1 = \frac{(1 + i4\sqrt{3})(1 - i4\sqrt{3})}{49},$$
$$1 = \frac{(11 + i4\sqrt{3})(11 - i4\sqrt{3})}{169}$$

Following the above analysis, one has two more sets of integer solutions to (1).

Method 2
Write (3) as

$$P^2 - 3v^2 = u^2 = u^2 * 1 \tag{10}$$

Assume

$$u = a^2 - 3b^2 \tag{11}$$

Write 1 as

$$1 = (2 + \sqrt{3}) * (2 - \sqrt{3}) \tag{12}$$

Substituting (11) & (12) in (10) and employing the method of factorization, consider

$$P + \sqrt{3}v = (2 + \sqrt{3})(a + \sqrt{3}b)^2 \tag{13}$$

from which we have

$$\left.\begin{aligned} P &= 2a^2 + 6b^2 + 6ab, \\ v &= a^2 + 3b^2 + 4ab \end{aligned}\right) \tag{14}$$

Substituting (11) & (14) in (2), the corresponding integer solutions to (1) are given by

$$\begin{aligned} x &= 4^n * (2a^2 + 4ab), \\ y &= 4^n * (-6b^2 - 4ab), \\ T &= 4^n * (2a^2 + 6b^2 + 6ab), \\ z &= (3a^2 - 3b^2 + 4ab) \\ w &= (a^2 - 9b^2 - 8ab) \end{aligned}$$

Note 4:
The integer 1 on the R.H.S. of (10) is also represented as below:

$$1 = (7 + 4\sqrt{3}) * (7 - 4\sqrt{3})$$

Following the above analysis, one has another set of integer solutions to (1).

Method 3

Introducing the linear transformation

$$x = u + v, y = u - v, z = 2u + v, w = 2u - v, u \neq v \neq 0 \tag{15}$$

in (1), it is written as

$$u^2 + 3v^2 = 4^{2n}T^2 \tag{16}$$

The process of obtaining integral solutions to (1) is illustrated below:

Set 1:

Substituting

$$v = 2ab, u = 3a^2 - b^2$$

in (16), we get

$$\left.\begin{array}{l} x = x(a,b) = 2ab + 3a^2 - b^2 \\ y = y(a,b) = -2ab + 3a^2 - b^2 \\ z = z(a,b) = 2ab + 6a^2 - 2b^2 \\ w = w(a,b) = -2ab + 6a^2 - 2b^2 \\ T = T(a,b) = \frac{1}{4^n}(3a^2 + b^2) \end{array}\right) \tag{17}$$

We choose a and b suitably so that the values of x, y, z, w and T are integers.
Replacing $a$by $4^n A$ and $b$by $4^n B$ in (17), the corresponding integer solutions of (1) in two parameters are

$$\begin{array}{l} x = x(A,B) = 4^{2n}(2AB + 3A^2 - B^2) \\ y = y(A,B) = 4^{2n}(-2AB + 3A^2 - B^2) \\ z = z(A,B) = 4^{2n}(2AB + 6A^2 - 2B^2) \\ w = w(A,B) = 4^{2n}(-2AB + 6A^2 - B^2) \\ T = T(A,B) = 4^n(3A^2 + B^2) \end{array}$$

Set 2:

Assume

$$T(a,b) = a^2 + 3b^2, a, b \neq 0 \tag{18}$$

Write 4 as

$$4 = (1 + i\sqrt{3})(1 - i\sqrt{3}) \tag{19}$$

Using (18) and (19) in (16) and employing the method of factorization, consider

$$u + i\sqrt{3}v = (1 + i\sqrt{3})^{2n}(a + i\sqrt{3}b)^2 \tag{20}$$

We write

$$(1 + i\sqrt{3})^{2n} = (\alpha + i\sqrt{3}\beta) \tag{21}$$

Using (21) in (20) and equating real and imaginary parts, we get

$$\begin{array}{l} u = u(a,b) = \alpha a^2 - 3\alpha b^2 - 6\beta ab \\ v = v(a,b) = \beta a^2 - 3\beta b^2 + 2\alpha ab \end{array}$$

Employing (15), the values of x, y, z and w are given by

$$\left.\begin{aligned} x &= x(a,b) = (\alpha+\beta)a^2 - 3(\alpha+\beta)b^2 + (\alpha-3\beta)2ab \\ y &= y(a,b) = (\alpha-\beta)a^2 + 3(\beta-\alpha)b^2 - (\alpha+3\beta)2ab \\ z &= z(a,b) = (2\alpha+\beta)a^2 - 3(2\alpha+\beta)b^2 + (\alpha-6\beta)2ab \\ w &= w(a,b) = (2\alpha-\beta)a^2 - 3(2\alpha-\beta)b^2 - (\alpha+6\beta)2ab \end{aligned}\right) \quad (22)$$

Thus, (18) and (22) represent the integer solutions to (1).
For simplicity, taking $n = 1$in (21), the corresponding integer solutions to (1) are found to be

$$\begin{aligned} x &= x(a,b) = -16ab \\ y &= y(a,b) = -4a^2 + 12b^2 - 8ab \\ z &= z(a,b) = -2a^2 + 6b^2 - 28ab \\ w &= w(a,b) = -6a^2 + 18b^2 - 20ab \\ T &= z(a,b) = a^2 + 3b^2 \end{aligned}$$

Note 5:
Observe that, in addition to (19), 4 may be taken as

$$4 = \frac{(11+i5\sqrt{3})(11-i5\sqrt{3})}{196}$$

The repetition of the above process leads to a different solution to (1).

3. Conclusion

An attempt has been made to obtain non-zero distinct integer solutions to the homogeneous bi-quadratic diophantine equation with five unknowns given by $2(x-y)(x^3+y^3) = 4^{2n}(z^2-w^2)T^2$. One may search for other sets of integer solutions to the considered equation as well as other choices of the fourth degree diophantine equations with multivariables.

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