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Generalized Three-Dimensional Affine Transformation in Mathematics

Neelam Singh
Department of Mathematics, Bundelkhand College, Jhansi, Uttar Pradesh, India

Corresponding Author: **Neelam Singh**

Abstract
Every branch of knowledge deals with sets of elements and relationship between these elements. In order to study these, it is useful to study various possible transformations on the elements and to determine which properties remains invariant under these transformations. Each science tries to define concepts, laws and invariance. In physics we consider atoms, electrons, protons, mass, momentum, force, energy, temperature and the structure of relationships among them is expressed by various laws such as Newton's Law of motion, Law of mass of conservation etc. In chemistry, we consider atomic and molecular structures and the transformations of inorganic and organic substances through chemical reactions. Similarly, Sociologists deal with social transformation, economists with economic transformations, and political scientists with political transformations and so on.

Keywords: Affine, Group, Mathematics, Model, Transformation, Structure, System

Introduction

Mathematics has been designed by man to provide intellectual models isomorphic to those in physical, chemical, biological and social sciences. It also deals with sets of elements, structures on these sets, transformations in them and their invariants. Clearly mathematics deals with relatively more abstract sets and structures than other [1].

The central idea in modern mathematics is that the study of a structure is best made in terms of transformations which preserve that structure. In fact, the application of mathematics to nature and society is also based on this idea [1]. All sciences deal with structure and in applying mathematics to them we attempt to find mathematical systems having structures similar to those in physical, biological and social sciences.

A physical, biological or social situation may be highly complicated, but at one time we are interested only in some particular aspect of it. We first find an appropriate transformation to convert this situation into a mathematical form which preserves the structure we are interested in. This process is called mathematical model making. The mathematical model is as good or as bad as its capacity to preserve the particular aspect of the structure we are considering. So, our problem is to find a transformation which is as close to reality as possible and which is capable of being handled by appropriate simplifying mathematical transformations. Thus, mathematical model making is an exercise in finding suitable physical mathematical transformations preserving the structures under study, keeping in mind the available mathematical transformations for simplifying the mathematical model. Once we have formed a mathematical model, we proceed to find mathematical transformations which will reduce it to a simpler form and preserve the structure in question or keep a certain property invariant.

Group of Transformations

A transformation T on a set X is a mapping of set X to itself, which is such that to every $x \in X,$ their exists unique $y \in X.$ We write $y = Tx$.

Here y is called image of x by transformation T and x is called pre image of y by transformation T. Transformation is a one-one, mapping of a set into itself. If G be the set of all transformations on X. The product of two transformations in G is also a transformation so we can say that closure property holds in G. For every $T_1, T_2, T_3 \in G,$

we have $(T_3T_2)T_1 = T_3(T_2T_1)$

Thus, associative law holds for multiplication of transformations in G.

The Transformation $I \in G,$ such that $Ix = x$

$$(IT)x = I(Tx) = Iy = y = Tx$$

&

$$(TI)x = T(Ix) = Tx = y = Tx$$

So that $IT = TI = T$

Thus I is identity transformation in G.

For every transformation T in G, there exists a transformation T^{-1} such that $T^{-1}y = x$, which is one-one, onto and $(TT^{-1})y = T(T^{-1}y) = Tx = y = Iy$

& $(T^{-1}T)x = T^{-1}(Tx) = T^{-1}y = x = Ix$

$$\therefore TT^{-1} = T^{-1}T = I$$

Thus, every transformation in G has an inverse transformation in G.

Since all postulates are satisfied in G, so set of all transformations on X forms a group for the operation of multiplication of transformations. Group G is not commutative in general.
Group of transformations which are likely to be of interest are those which preserve some geometrical property such as incidence, co linearity, concurrence, parallelism, distance, order, angle, direction, area, co circularity, congruence, similarity, continuity, cross ratio of points etc. Most of the modern curriculum reform projects in school mathematics [2] have given an important place to transformation geometry, especially to the study of isometric (distance preserving) transformations. Several new undergraduate courses also contain a discussion of metric geometry over affine space [3, 4]. In present paper concept of affine transformation

Affine Transformations: Consider the transformation $x' = ax + by + cz + d,\ y' = a_1x + b_1y + c_1z + d_1, z' = a_2x + b_2y + c_2z + d_2$………………………..(1) from the set of all ordered pairs (x, y, z) of real numbers to the set of ordered pairs (x', y', z') of real numbers. For this transformation to be one-one, we should have a unique (x, y, z) to correspond to a given point $(x'.y', z')$.

Let $\Delta = \begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \neq 0$

Solving (1) for x, y, z by Cramer's Rule; We get

$$x = \frac{\begin{vmatrix} x'-d & b & c \\ y'-d_1 & b_1 & c_1 \\ z'-d_2 & b_2 & c_2 \end{vmatrix}}{\Delta} = \frac{(x'-d)(b_1c_2-b_2c_1)-(y'-d_1)(bc_2-b_2c)+(z'-d_2)(bc_1-b_1c)}{\Delta}$$

$$= \frac{x'(b_1c_2-b_2c_1)+y'(b_2c-bc_2)+z'(bc_1-b_1c)-\{d(b_1c_2-b_2c_1)+d_1(b_2c-bc_2)+d_2(bc_1-b_1c)\}}{\Delta};$$

$$y = \frac{\begin{vmatrix} a & x'-d & c \\ a_1 & y'-d_1 & c_1 \\ a_2 & z'-d_2 & c_2 \end{vmatrix}}{\Delta} = \frac{-(x'-d)(a_1c_2-a_2c_1)+(y'-d_1)(ac_2-a_2c)-(z'-d_2)(ac_1-a_1c)}{\Delta}$$

$$= \frac{x'(a_2c_1-a_1c_2)+y'(ac_2-a_2c)+z'(a_1c-ac_1)-\{d(a_2c_1-a_1c_2)+d_1(ac_2-a_2c)+d_2(a_1c-ac_1)\}}{\Delta}$$

And $z = \dfrac{\begin{vmatrix} a & b & x'-d \\ a_1 & b_1 & y'-d_1 \\ a_2 & b_2 & z'-d_2 \end{vmatrix}}{\Delta} = \dfrac{(x'-d)(a_1b_2-a_2b_1)-(y'-d_1)(ab_2-a_2b)+(z'-d_2)(ab_1-a_1b)}{\Delta}$

$$= \frac{x'(a_1b_2-a_2b_1)+y'(a_2b-ab_2)+z'(ab_1-a_1b)-\{d(a_1b_2-a_2b_1)+d_1(a_2b-ab_2)+d_2(ab_1-a_1b)\}}{\Delta}$$

Or

$$x = \frac{(b_1c_2-b_2c_1)}{\Delta}x' + \frac{(b_2c-bc_2)}{\Delta}y' + \frac{(bc_1-b_1c)}{\Delta}z' - \frac{\{d(b_1c_2-b_2c_1)+d_1(b_2c-bc_2)+d_2(bc_1-b_1c)\}}{\Delta},$$

$$y = \frac{(a_2c_1-a_1c_2)}{\Delta}x' + \frac{(ac_2-a_2c)}{\Delta}y' + \frac{(a_1c-ac_1)}{\Delta}z' - \frac{\{d(a_2c_1-a_1c_2)+d_1(ac_2-a_2c)+d_2(a_1c-ac_1)\}}{\Delta},$$

$$z = \frac{(a_1b_2-a_2b_1)}{\Delta}x' + \frac{(a_2b-ab_2)}{\Delta}y' + \frac{(ab_1-a_1b)}{\Delta}z' - \frac{\{d(a_1b_2-a_2b_1)+d_1(a_2b-ab_2)+d_2(ab_1-a_1b)\}}{\Delta}, \qquad (2)$$

Thus, to given point (x',y',z') there corresponds a unique point (x,y,z) if $\Delta=\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \neq 0$ (3)

If Δ' be the determinant of the coefficients of x', y', z'; Then

$$\Delta' = \begin{vmatrix} \frac{(b_1c_2-b_2c_1)}{\Delta} & \frac{(b_2c-bc_2)}{\Delta} & \frac{(bc_1-b_1c)}{\Delta} \\ \frac{(a_2c_1-a_1c_2)}{\Delta} & \frac{(ac_2-a_2c)}{\Delta} & \frac{(a_1c-ac_1)}{\Delta} \\ \frac{(a_1b_2-a_2b_1)}{\Delta} & \frac{(a_2b-ab_2)}{\Delta} & \frac{(ab_1-a_1b)}{\Delta} \end{vmatrix}$$

Multiplying first row by a, second row by b and third row by c, and adding we get

$$\Delta' = \frac{1}{abc}\begin{vmatrix} \frac{\Delta}{\Delta} & 0 & 0 \\ \frac{b(a_2c_1-a_1c_2)}{\Delta} & \frac{b(ac_2-a_2c)}{\Delta} & \frac{b(a_1c-ac_1)}{\Delta} \\ \frac{c(a_1b_2-a_2b_1)}{\Delta} & \frac{c(a_2b-ab_2)}{\Delta} & \frac{c(ab_1-a_1b)}{\Delta} \end{vmatrix} = \frac{1}{\Delta^2 a}[(ac_2-a_2c)(ab_1-a_1b)-(a_1c-ac_1)(a_2b-ab_2)]$$

$$= \frac{1}{\Delta^2 a}[a^2(b_1c_2-b_2c_1)+ab(a_2c_1-a_1c_2)+ac(a_1b_2-a_2b_1)] = \frac{a\Delta}{\Delta^2 a} = \frac{1}{\Delta}$$

i.e., $\Delta' = \frac{1}{\Delta}$

Therefore (1) gives a one -one mapping if (3) is satisfied, we can say this transformation as generalized affine transformation in three dimensional spaces and the quantity

$$\Delta=\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = [a(b_1c_2-b_2c_1)+b(a_2c_1-a_1c_2)+c(a_1b_2-a_2b_1)]$$

is known as the determinant of the transformation.

Hence generalized affine transformation in three dimensions is given by $x' = ax+by+cz+d,\ y' = a_1x+b_1y+c_1z+d_1, z' = a_2x+b_2y+c_2z+d_2,\ \Delta \neq 0$ (4)

Now (2) will determine generalized affine transformation in three-dimension if

$$\Delta' = \begin{vmatrix} \frac{(b_1c_2-b_2c_1)}{\Delta} & \frac{(b_2c-bc_2)}{\Delta} & \frac{(bc_1-b_1c)}{\Delta} \\ \frac{(a_2c_1-a_1c_2)}{\Delta} & \frac{(ac_2-a_2c)}{\Delta} & \frac{(a_1c-ac_1)}{\Delta} \\ \frac{(a_1b_2-a_2b_1)}{\Delta} & \frac{(a_2b-ab_2)}{\Delta} & \frac{(ab_1-a_1b)}{\Delta} \end{vmatrix} = \frac{1}{\Delta} \neq 0$$

i.e., $\Delta^{-1} \neq 0$ which is satisfied because $\Delta \neq 0$. Thus Equations (2) also determine generalized affine transformation in three dimensions.

When transformation (1) maps a point P to a point Q, Then transformation (2)

sends Q back to P. So, we can say transformation (2) is transformation inverse to transformation (1).Hence every generalized affine transformation in three dimension has an inverse transformation which itself is generalized affine transformation in three dimension. Determinant of the coefficients of inverse transformation is reciprocal of the determinant of the original transformation (1).

The Product of two transformations

$$x' = ax+by+cz+d,\ y' = a_1x+b_1y+c_1z+d_1, z' = a_2x+b_2y+c_2z+d_2;$$

&

$$x'' = \frac{(b_1c_2-b_2c_1)}{\Delta}x' + \frac{(b_2c-bc_2)}{\Delta}y' + \frac{(bc_1-b_1c)}{\Delta}z' - \frac{\{d(b_1c_2-b_2c_1)+d_1(b_2c-bc_2)+d_2(bc_1-b_1c)\}}{\Delta},$$

$$y'' = \frac{(a_2c_1-a_1c_2)}{\Delta}x' + \frac{(ac_2-a_2c)}{\Delta}y' + \frac{(a_1c-ac_1)}{\Delta}z' - \frac{\{d(a_2c_1-a_1c_2)+d_1(ac_2-a_2c)+d_2(a_1c-ac_1)\}}{\Delta},$$

$$z'' = \frac{(a_1b_2-a_2b_1)}{\Delta}x' + \frac{(a_2b-ab_2)}{\Delta}y' + \frac{(ab_1-a_1b)}{\Delta}z' - \frac{\{d(a_1b_2-a_2b_1)+d_1(a_2b-ab_2)+d_2(ab_1-a_1b)\}}{\Delta},\ldots$$

Where $\Delta = \begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \neq 0$ is given by

$$x'' = \frac{(b_1c_2-b_2c_1)}{\Delta}(ax+by+cz+d) + \frac{(b_2c-bc_2)}{\Delta}(a_1x+b_1y+c_1z+d_1) + \frac{(bc_1-b_1c)}{\Delta}(a_2x+b_2y+c_2z+d_2) - \frac{\{d(b_1c_2-b_2c_1)+d_1(b_2c-bc_2)+d_2(bc_1-b_1c)\}}{\Delta} = \frac{\Delta}{\Delta}x + 0y + 0z + 0 = x,$$

$$y'' = \frac{(a_2c_1-a_1c_2)}{\Delta}(ax+by+cz+d) + \frac{(ac_2-a_2c)}{\Delta}(a_1x+b_1y+c_1z+d_1)$$

$$+\frac{(a_1c-ac_1)}{\Delta}(a_2x+b_2y+c_2z+d_2) - \frac{\{d(a_2c_1-a_1c_2)+d_1(ac_2-a_2c)+d_2(a_1c-ac_1)\}}{\Delta}$$

$$= 0x + \frac{\Delta}{\Delta}y + 0z + 0 = y,$$

$$z'' = \frac{(a_1b_2-a_2b_1)}{\Delta}(ax+by+cz+d) + \frac{(a_2b-ab_2)}{\Delta}(a_1x+b_1y+c_1z+d_1)$$

$$+\frac{(ab_1-a_1b)}{\Delta}(a_2x+b_2y+c_2z+d_2) - \frac{\{d(a_1b_2-a_2b_1)+d_1(a_2b-ab_2)+d_2(ab_1-a_1b)\}}{\Delta}$$

$$= 0x + 0y + \frac{\Delta}{\Delta}z + 0 = z$$

i.e., $x'' = x,\ y'' = y, z'' = z$

This is identity transformation and is also generalized affine transformation in three dimensions. Thus each of the transformation (1) and (2) is the inverse of the other.
Now we show that set of all transformations of form (1) forms a group.

1. The product of two generalized affine transformations in three dimensions $x' = ax + by + cz + d$

$$y' = a_1x + b_1y + c_1z + d_1$$

$$z' = a_2x + b_2y + c_2z + d_2$$

With $\Delta = \begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \neq 0$ (i)

&

$$x'' = a'x' + b'y' + c'z' + d'$$

$$y'' = a_1'x' + b_1'y' + c_1'z' + d_1'$$

$$z'' = a_2'x' + b_2'y' + c_2'z' + d_2'$$

With $\Delta' = \begin{vmatrix} a' & b' & c' \\ a_1' & b_1' & c_1' \\ a_2' & b_2' & c_2' \end{vmatrix} \neq 0$ (ii)

; is given by

$$x'' = a'(ax+by+cz+d) + b'(a_1x+b_1y+c_1z+d_1) + c'(a_2x+b_2y+c_2z+d_2) + d'$$

$$y'' = a_1'(ax+by+cz+d) + b_1'(a_1x+b_1y+c_1z+d_1) + c_1'(a_2x+b_2y+c_2z+d_2) + d_1'$$

$$z'' = a_2'(ax+by+cz+d) + b_2'(a_1x+b_1y+c_1z+d_1) + c_2'(a_2x+b_2y+c_2z+d_2) + d_2'$$

i.e.,

$$x'' = (a'a+b'a_1+c'a_2)x + (a'b+b'b_1+c'b_2)y + (a'c+b'c_1+c'c_2)z + (a'd+b'd_1+c'd_2+d')$$

$$y'' = (a_1'a+b_1'a_1+c_1'a_2)x + (a_1'b+b_1'b_1+c_1'b_2)y + (a_1'c+b_1'c_1+c_1'c_2)z + (a_1'd+b_1'd_1+c_1'd_2+d_1')$$

$$z'' = (a_2'a+b_2'a_1+c_2'a_2)x + (a_2'b+b_2'b_1+c_2'b_2)y + (a_2'c+b_2'c_1+c_2'c_2)z + (a_2'd+b_2'd_1+c_2'd_2+d_2') \quad \text{(iii)}$$

Determinant of the coefficients of x, y, z is given by

$$\Delta'' = \begin{vmatrix} a'a+b'a_1+c'a_2 & a'b+b'b_1+c'b_2 & a'c+b'c_1+c'c_2 \\ a_1'a+b_1'a_1+c_1'a_2 & a_1'b+b_1'b_1+c_1'b_2 & a_1'c+b_1'c_1+c_1'c_2 \\ a_2'a+b_2'a_1+c_2'a_2 & a_2'b+b_2'b_1+c_2'b_2 & a_2'c+b_2'c_1+c_2'c_2 \end{vmatrix}$$

$$= \begin{vmatrix} a'a & a'b+b'b_1+c'b_2 & a'c+b'c_1+c'c_2 \\ a_1'a & a_1'b+b_1'b_1+c_1'b_2 & a_1'c+b_1'c_1+c_1'c_2 \\ a_2'a & a_2'b+b_2'b_1+c_2'b_2 & a_2'c+b_2'c_1+c_2'c_2 \end{vmatrix}$$

$$+ \begin{vmatrix} b'a_1 & a'b+b'b_1+c'b_2 & a'c+b'c_1+c'c_2 \\ b_1'a_1 & a_1'b+b_1'b_1+c_1'b_2 & a_1'c+b_1'c_1+c_1'c_2 \\ b_2'a_1 & a_2'b+b_2'b_1+c_2'b_2 & a_2'c+b_2'c_1+c_2'c_2 \end{vmatrix}$$

$$+ \begin{vmatrix} c'a_2 & a'b+b'b_1+c'b_2 & a'c+b'c_1+c'c_2 \\ c_1'a_2 & a_1'b+b_1'b_1+c_1'b_2 & a_1'c+b_1'c_1+c_1'c_2 \\ c_2'a_2 & a_2'b+b_2'b_1+c_2'b_2 & a_2'c+b_2'c_1+c_2'c_2 \end{vmatrix}$$

$$= \begin{vmatrix} a'a & a'b & a'c+b'c_1+c'c_2 \\ a_1'a & a_1'b & a_1'c+b_1'c_1+c_1'c_2 \\ a_2'a & a_2'b & a_2'c+b_2'c_1+c_2'c_2 \end{vmatrix} + \begin{vmatrix} a'a & b'b_1 & a'c+b'c_1+c'c_2 \\ a_1'a & b_1'b_1 & a_1'c+b_1'c_1+c_1'c_2 \\ a_2'a & b_2'b_1 & a_2'c+b_2'c_1+c_2'c_2 \end{vmatrix} + \begin{vmatrix} a'a & c'b_2 & a'c+b'c_1+c'c_2 \\ a_1'a & c_1'b_2 & a_1'c+b_1'c_1+c_1'c_2 \\ a_2'a & c_2'b_2 & a_2'c+b_2'c_1+c_2'c_2 \end{vmatrix}$$

$$+ \begin{vmatrix} b'a_1 & a'b & a'c+b'c_1+c'c_2 \\ b_1'a_1 & a_1'b & a_1'c+b_1'c_1+c_1'c_2 \\ b_2'a_1 & a_2'b & a_2'c+b_2'c_1+c_2'c_2 \end{vmatrix} + \begin{vmatrix} b'a_1 & b'b_1 & a'c+b'c_1+c'c_2 \\ b_1'a_1 & b_1'b_1 & a_1'c+b_1'c_1+c_1'c_2 \\ b_2'a_1 & b_2'b_1 & a_2'c+b_2'c_1+c_2'c_2 \end{vmatrix}$$

$$+ \begin{vmatrix} b'a_1 & c'b_2 & a'c+b'c_1+c'c_2 \\ b_1'a_1 & c_1'b_2 & a_1'c+b_1'c_1+c_1'c_2 \\ b_2'a_1 & c_2'b_2 & a_2'c+b_2'c_1+c_2'c_2 \end{vmatrix} + \begin{vmatrix} c'a_2 & a'b & a'c+b'c_1+c'c_2 \\ c_1'a_2 & a_1'b & a_1'c+b_1'c_1+c_1'c_2 \\ c_2'a_2 & a_2'b & a_2'c+b_2'c_1+c_2'c_2 \end{vmatrix}$$

$$+ \begin{vmatrix} c'a_2 & b'b_1 & a'c+b'c_1+c'c_2 \\ c_1'a_2 & b_1'b_1 & a_1'c+b_1'c_1+c_1'c_2 \\ c_2'a_2 & b_2'b_1 & a_2'c+b_2'c_1+c_2'c_2 \end{vmatrix} + \begin{vmatrix} c'a_2 & c'b_2 & a'c+b'c_1+c'c_2 \\ c_1'a_2 & c_1'b_2 & a_1'c+b_1'c_1+c_1'c_2 \\ c_2'a_2 & c_2'b_2 & a_2'c+b_2'c_1+c_2'c_2 \end{vmatrix}$$

$$\begin{aligned} = 0 &+ \begin{vmatrix} a'a & b'b_1 & b'c_1 \\ a'_1a & b'_1b_1 & b'_1c_1 \\ a_2'a & b_2'b_1 & b_2'c_1 \end{vmatrix} + \begin{vmatrix} a'a & b'b_1 & c'c_2 \\ a'_1a & b'_1b_1 & c'_1c_2 \\ a_2'a & b_2'b_1 & c_2'c_2 \end{vmatrix} + \begin{vmatrix} a'a & c'b_2 & a'c \\ a'_1a & c'_1b_2 & a'_1c \\ a_2'a & c_2'b_2 & a_2'c \end{vmatrix} + \begin{vmatrix} a'a & c'b_2 & b'c_1 \\ a'_1a & c'_1b_2 & b'_1c_1 \\ a_2'a & c_2'b_2 & b_2'c_1 \end{vmatrix} + \begin{vmatrix} a'a & c'b_2 & c'c_2 \\ a'_1a & c'_1b_2 & c'_1c_2 \\ a_2'a & c_2'b_2 & c_2'c_2 \end{vmatrix} \\ &+ \begin{vmatrix} b'a_1 & a'b & a'c \\ b'_1a_1 & a'_1b & a'_1c \\ b_2'a_1 & a_2'b & a_2'c \end{vmatrix} + \begin{vmatrix} b'a_1 & a'b & b'c_1 \\ b'_1a_1 & a'_1b & b'_1c_1 \\ b_2'a_1 & a_2'b & b_2'c_1 \end{vmatrix} + \begin{vmatrix} b'a_1 & a'b & c'c_2 \\ b'_1a_1 & a'_1b & c'_1c_2 \\ b_2'a_1 & a_2'b & c_2'c_2 \end{vmatrix} + 0 + \begin{vmatrix} b'a_1 & c'b_2 & a'c \\ b'_1a_1 & c'_1b_2 & a'_1c \\ b_2'a_1 & c_2'b_2 & a_2'c \end{vmatrix} \\ &+ \begin{vmatrix} b'a_1 & c'b_2 & b'c_1 \\ b'_1a_1 & c'_1b_2 & b'_1c_1 \\ b_2'a_1 & c_2'b_2 & b_2'c_1 \end{vmatrix} + \begin{vmatrix} b'a_1 & c'b_2 & c'c_2 \\ b'_1a_1 & c'_1b_2 & c'_1c_2 \\ b_2'a_1 & c_2'b_2 & c_2'c_2 \end{vmatrix} + \begin{vmatrix} c'a_2 & a'b & a'c \\ c'_1a_2 & a'_1b & a'_1c \\ c_2'a_2 & a_2'b & a_2'c \end{vmatrix} + \begin{vmatrix} c'a_2 & a'b & b'c_1 \\ c'_1a_2 & a'_1b & b'_1c_1 \\ c_2'a_2 & a_2'b & b_2'c_1 \end{vmatrix} \\ &+ \begin{vmatrix} c'a_2 & a'b & c'c_2 \\ c'_1a_2 & a'_1b & c'_1c_2 \\ c_2'a_2 & a_2'b & c_2'c_2 \end{vmatrix} + \begin{vmatrix} c'a_2 & b'b_1 & a'c \\ c'_1a_2 & b'_1b_1 & a'_1c \\ c_2'a_2 & b_2'b_1 & a_2'c \end{vmatrix} + \begin{vmatrix} c'a_2 & b'b_1 & b'c_1 \\ c'_1a_2 & b'_1b_1 & b'_1c_1 \\ c_2'a_2 & b_2'b_1 & b_2'c_1 \end{vmatrix} + \begin{vmatrix} c'a_2 & b'b_1 & c'c_2 \\ c'_1a_2 & b'_1b_1 & c'_1c_2 \\ c_2'a_2 & b_2'b_1 & c_2'c_2 \end{vmatrix} + 0 \end{aligned}$$

$$= 0 + 0 + ab_1c_2\Delta' + 0 - ab_2c_1\Delta' + 0 + 0 + 0 - ba_1c_2\Delta + 0 + ca_1b_2\Delta' + 0 + 0 + 0 + ba_2c_1\Delta' + 0 - ca_2b_1\Delta' + 0 + 0 + 0$$

$$= \Delta'[a(b_1c_2 - b_2c_1) + b(a_2c_1 - a_1c_2) + c(a_1b_2 - a_2b_1)] = \Delta'\Delta$$

$$\Delta'' = \Delta'\Delta$$

This will be generalized affine transformation if $\Delta'' \neq 0$ i.e., $\Delta'\Delta \neq 0$,

$\because \Delta \neq 0$ & $\Delta' \neq 0$, so condition $\Delta'' \neq 0$ is satisfied.

Thus, product of two generalized affine transformations in three dimensions is also generalized affine transformation in three dimensions. Thus, closure property holds in the set of affine transformation for multiplication of transformation as operation.

2. Let first transformation of (1) be T_1 and second be T_2.Then transformation (iii) is denoted by T_2T_1. Let T_3 be a third transformation defined by

$$x''' = a''x'' + b''y'' + c''z'' + d''$$

$$y''' = a_1''x'' + b_1''y'' + c_1''z'' + d_1''$$

$$z''' = a_2''x'' + b_2''y'' + c_2''z'' + d_2''$$

With
$$\Delta''' = \begin{vmatrix} a'' & b'' & c'' \\ a_1'' & b_1'' & c_1'' \\ a_2'' & b_2'' & c_2'' \end{vmatrix} \neq 0$$

Then we have $T_3(T_2T_1) = (T_3T_2)T_1$

$\therefore$ Associative law holds for multiplication of transformation.

3. There exists generalized transformation I defined by

$$x' = x, y' = y, z' = z$$

For every transformation T,

$$TI = IT = T$$

$\therefore$ Set of generalized affine transformation possesses identity element.

4. For every generalized affine transformation T, there exists an inverse generalized transformation T^{-1} which is such that $TT^{-1} = T^{-1}T = I$

Thus, set of all generalized affine transformation forms a group.

Properties of Three dimensional Generalized Affine Transformations:

1. Generalized three-dimensional affine transformation maps plane onto planes.

Let the equation of a plane be given by $Ax + By + Cz + D = 0$
Then transformation (2) maps it onto

$$A\left(\frac{(b_1c_2 - b_2c_1)}{\Delta}x' + \frac{(b_2c - bc_2)}{\Delta}y' + \frac{(bc_1 - b_1c)}{\Delta}z' - \frac{\{d(b_1c_2 - b_2c_1) + d_1(b_2c - bc_2) + d_2(bc_1 - b_1c)\}}{\Delta}\right)$$

$$+B\left(\frac{(a_2c_1 - a_1c_2)}{\Delta}x' + \frac{(ac_2 - a_2c)}{\Delta}y' + \frac{(a_1c - ac_1)}{\Delta}z' - \frac{\{d(a_2c_1 - a_1c_2) + d_1(ac_2 - a_2c) + d_2(a_1c - ac_1)\}}{\Delta}\right)$$

$$+C\left(\frac{(a_1b_2 - a_2b_1)}{\Delta}x' + \frac{(a_2b - ab_2)}{\Delta}y' + \frac{(ab_1 - a_1b)}{\Delta}z' - \frac{\{d(a_1b_2 - a_2b_1) + d_1(a_2b - ab_2) + d_2(ab_1 - a_1b)\}}{\Delta}\right) + D = 0$$

Clearly above equation represent plane since it is linear in x', y', z'.
2. Similarly generalized three-dimensional affine transformation maps a family of parallel planes onto another family of parallel planes.
3. Generalized three-dimensional affine transformation maps three-dimensional triangle ABC onto a triangle A'B'C', the points inside (outside) triangle ABC are mapped onto points inside (outside) triangle A'B'C' and centroid of triangle ABC is mapped onto centroid of triangle A'B'C'.
4. Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ be mapped onto $A'(x_1', y_1', z_1')$, $B'(x_2', y_2', z_2')$ respectively.

Then $(AB)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

And $(A'B')^2 = (x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2$

$$= (ax_2 + by_2 + cz_2 + d - ax_1 - by_1 - cz_1 - d)^2 + (a_1x_2 + b_1y_2 + c_1z_2 + d_1 - a_1x_1 - b_1y_1 - c_1z_1 - d_1)^2 + (a_2x_2 + b_2y_2 + c_2z_2 + d_2 - a_2x_1 - b_2y_1 - c_2z_1 - d_2)^2$$

$$= (a^2 + a_1^2 + a_2^2)(x_2 - x_1)^2 + (b^2 + b_1^2 + b_2^2)(y_2 - y_1)^2 + (c^2 + c_1^2 + c_2^2)(z_2 - z_1)^2 + 2(ab + a_1b_1 + a_2b_2)(x_2 - x_1)(y_2 - y_1) + 2(bc + b_1c_1 + b_2c_2)(y_2 - y_1)(z_2 - z_1) + 2(ca + c_1a_1 + c_2a_2)(z_2 - z_1)(x_2 - x_1)$$

In general, $A'B' \neq AB$, therefore, in general Generalized three-dimensional affine transformation do not preserve distances between points.

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